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Hyperbolicity and integral expression of the Lyapunov exponents for linear cocycles ☆

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Abstract

Consider in this paper a linear skew-product system

$$(\theta, \Theta): \mathbb{T} \times W \times \mathbb{R}^n \to W \times \mathbb{R}^n; \quad (t, w, x) \mapsto \big(t.w, \Theta(t, w) \cdot x\big)$$

where $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, and $\theta : (t, w) \mapsto t.w$ is a topological dynamical system on a compact metrizable space W, and where $\Theta(t, w) \in GL(n, \mathbb{R})$ satisfies the cocycle condition based on θ and is continuously differentiable in t if $\mathbb{T} = \mathbb{R}$. We show that 'semi λ-exponential dichotomy' of (θ, Θ) implies 'λ-exponential dichotomy.' Precisely, if Θ has no Lyapunov exponent λ and is almost uniformly λ-contracting along the λ-stable direction $\mathbb{E}^s(w; \lambda)$ and if $\dim \mathbb{E}^s(w; \lambda)$ is constant a.e., then Θ is almost λ-exponentially dichotomous. To prove this, we first use Liao's spectrum theorem, which gives integral expression of the Lyapunov exponents, and then use the semi-uniform ergodic theorem by Sturman and Stark, which allows one to derive uniform estimates from nonuniform ones. As a consequence, we obtain the open-and-dense hyperbolicity of eventual $GL_+(2, \mathbb{R})$-cocycles based on a uniquely ergodic endomorphism, and of $GL(2, \mathbb{R})$-cocycles based on a uniquely ergodic equi-continuous endomorphism, respectively.

On the other hand, in the sense of C^0-topology we obtain the density of $SL(2, \mathbb{R})$-cocycles having positive Lyapunov exponent based on a minimal subshift satisfying the Boshernitzan condition.

MSC: primary 37D20; secondary 37D25, 37H15, 37C10

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1. Introduction

Hyperbolicity (or exponential dichotomy) is a very important condition in the study of differentiable dynamical systems [40,32]. So, it is always significant to judge whether a smooth system has hyperbolic (or exponentially dichotomous) behaviors or not. One of the important ways is to obtain hyperbolicity from nonuniform hyperbolicity. There already, however, exist many examples which show that nonuniform hyperbolicity is strictly weaker than hyperbolicity; for example, [6, §8] and [47]. Naturally, we ask:

Question 1. When does nonuniform hyperbolicity imply uniform hyperbolicity?

Recently, this topic becomes more and more interesting; see [44,23,5,12,41,46,13,14,20,19] amongst others. In this paper, we consider a linear skew-product system

$$(\theta, \Theta): \mathbb{T} \times W \times \mathbb{R}^n \to W \times \mathbb{R}^n; \quad (t, w, x) \mapsto \big(t.w, \Theta(t, w) \cdot x\big) \tag{1.1}$$

where $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, and $\theta : (t, w) \mapsto t.w$ is a topological dynamical system on a compact metrizable space W, and where $\Theta : \mathbb{T} \times W \to GL(n, \mathbb{R})$ satisfies the cocycle property

$$\Theta(s + t, w) \cdot x = \Theta(s, t.w) \circ \Theta(t, w) \cdot x$$

for any $s, t \in \mathbb{T}$ and for any $x \in \mathbb{R}^n$, and is continuously differentiable with respect to t if $\mathbb{T} = \mathbb{R}$.

We denote by $\mathcal{M}_{\mathrm{erg}}(W, \theta)$ the set of all ergodic θ-invariant Borel probability measures supported on W, endowed with the usual weak $*$-topology. Since (W, θ) is a compact topological dynamical system, this set is nonempty from the standard ergodic theory [35, Chapter VI].

By virtue of the Oseledets multiplicative ergodic theorem [37,28,31], to a.e. $w \in W$, there exists a *measurable* direct decomposition of $\mathbb{R}^n$ into subspaces

$$w \mapsto \mathbb{E}_1(w) \oplus \cdots \oplus \mathbb{E}_{\delta(w)}(w) \tag{1.2}$$

and $\delta(w)$ real numbers, called the Lyapunov exponents of (θ, Θ) at point w,

$$\lambda_1(w) < \cdots < \lambda_{\delta(w)}(w) \tag{1.3a}$$

such that

$$\lambda_i(w) = \lim_{t \to \pm\infty} \frac{1}{t} \log\big\|\Theta(t, w) \cdot x\big\| \quad \forall x \in \mathbb{E}_i(w) \setminus \{0\} \tag{1.3b}$$

and λ_i is θ-invariant and $\Theta(t, w) \cdot \mathbb{E}_i(w) = \mathbb{E}_i(t.w)$ for $1 \leqslant i \leqslant \delta(w)$.

1.1. Criterion of hyperbolicity

Given $\hat{\lambda} \in \mathbb{R}$. If

$$\hat{\lambda} \neq \lambda_i(w), \quad 1 \leqslant i \leqslant \delta(w), \tag{1.4a}$$

holds for a.e. $w \in W$, let us write

$$\mathbb{E}^s(w; \hat{\lambda}) = \bigoplus_{\lambda_i(w) < \hat{\lambda}} \mathbb{E}_i(w) \quad \text{and} \quad \mathbb{E}^u(w; \hat{\lambda}) = \bigoplus_{\lambda_i(w) > \hat{\lambda}} \mathbb{E}_i(w), \tag{1.4b}$$

called the $\hat{\lambda}$-*stable* and $\hat{\lambda}$-*unstable directions* of (θ, Θ) at w, respectively. Moreover we call

$$\operatorname{Index}(\hat{\lambda}; w) := \dim \mathbb{E}^s(w; \hat{\lambda}) \tag{1.4c}$$

the $\hat{\lambda}$-*index* of (θ, Θ) at w. In the special nonuniformly hyperbolic case $\hat{\lambda} = 0$, as usual we simply write

$$\mathbb{E}^s(w) = \bigoplus_{\lambda_i(w)<0} \mathbb{E}_i(w) \quad \text{and} \quad \mathbb{E}^u(w) = \bigoplus_{\lambda_i(w)>0} \mathbb{E}_i(w). \tag{1.5}$$

Notice here that $w \mapsto \mathbb{E}^s(w; \hat{\lambda}) \oplus \mathbb{E}^u(w; \hat{\lambda})$ is measurable, not necessarily continuous. In the future, we let $\mathbb{T}^+ = \{t \in \mathbb{T} \mid t \geqslant 0\}$ and $\mathbb{T}^- = \{t \in \mathbb{T} \mid t \leqslant 0\}$.

For the simplicity, we first introduce the notations 'semi-exponential dichotomy' and 'semi-hyperbolicity,' which are conceptually weaker than dominated splitting condition.

Definition 1. Given $\hat{\lambda} \in \mathbb{R}$. (θ, Θ) is called to be *semi* $\hat{\lambda}$-*exponentially dichotomous*, provided that $\lambda_i(w) \neq \hat{\lambda}$ for $1 \leqslant i \leqslant \delta(w)$ and for a.e. $w \in W$, and that Θ is almost uniformly $\hat{\lambda}$-expanding (resp. $\hat{\lambda}$-contracting) along $\mathbb{E}^u(w; \hat{\lambda})$ (resp. $\mathbb{E}^s(w; \hat{\lambda})$); that is to say, there are constants $\varrho' > 0$ and $C' > 0$, which both are independent of w, such that

$$\bigl\|\Theta(t, w) \cdot x\bigr\| \geqslant C' e^{t(\hat{\lambda}+\varrho')} \|x\| \quad \forall x \in \mathbb{E}^u(w; \hat{\lambda}) \tag{1.6a}$$

$$\bigl(\text{resp. } \bigl\|\Theta(t, w) \cdot x\bigr\| \leqslant C' e^{t(\hat{\lambda}-\varrho')} \|x\| \quad \forall x \in \mathbb{E}^s(w; \hat{\lambda})\bigr) \tag{1.6b}$$

for all $t \in \mathbb{T}^+$ and for a.e. $w \in W$. If (1.6a) and (1.6b) hold simultaneously for a.e. $w \in W$, we say (θ, Θ) to be *almost* $\hat{\lambda}$-*exponentially dichotomous*.

Particularly, (θ, Θ) is called *semi-hyperbolic*, provided that (θ, Θ) is semi 0-exponentially dichotomous. That is to say, $\lambda_i(w) \neq 0$ for $1 \leqslant i \leqslant \delta(w)$ and for a.e. $w \in W$, and that Θ is almost uniformly expanding (resp. contracting) along $\mathbb{E}^u(w)$ (resp. $\mathbb{E}^s(w)$); i.e., there are constants $\varrho' > 0$ and $C' > 0$ such that

$$\bigl\|\Theta(t, w) \cdot x\bigr\| \geqslant C' e^{t\varrho'} \|x\| \quad \forall x \in \mathbb{E}^u(w) \tag{1.7a}$$

$$\bigl(\text{resp. } \bigl\|\Theta(t, w) \cdot x\bigr\| \leqslant C' e^{-t\varrho'} \|x\| \quad \forall x \in \mathbb{E}^s(w)\bigr) \tag{1.7b}$$

for all $t \in \mathbb{T}^+$ and for a.e. $w \in W$. If (1.7a) and (1.7b) hold simultaneously for a.e. $w \in W$, we call (θ, Θ) to be *almost hyperbolic*.

Here and in what follows, 'a.e.' means relative to all $\nu \in \mathcal{M}_{\text{erg}}(W, \theta)$ unless an explicit measure ν is given and write 'ν-a.e.' in this case.

To well understand the semi hyperbolicity, let us consider first a more strong condition: If there are two constants $\varrho' > 0$ and $C' > 0$ such that

$$\bigl\|\Theta(t, w) \cdot x\bigr\| \geqslant C' e^{t\varrho'} \|x\| \quad \forall x \in \mathbb{E}^u(w) \tag{1.8a}$$

and

$$\bigl\|\Theta(t, w) \cdot x\bigr\| \leqslant C' e^{-t\varrho'} \|x\| \quad \forall x \in \mathbb{E}^s(w) \tag{1.8b}$$

for all $t \in \mathbb{T}^+$ and for $w \in \Gamma$ (not necessarily closed), then by the uniformity conditions (1.8a) and (1.8b) we easily obtain the continuous continuation of $w \mapsto \mathbb{E}^u(w) \oplus \mathbb{E}^s(w)$ from Γ onto its closure $\overline{\Gamma}$. However, if (1.8a) and (1.8b) need not hold simultaneously, the case becomes quite hard, because we cannot easily obtain

$$\lim_{i\to\infty} \mathbb{E}^s(w_i) = \lim_{i\to\infty} \mathbb{E}^s\big(w_i'\big)$$
$$\Big(\text{resp. } \lim_{i\to\infty} \mathbb{E}^u(w_i) = \lim_{i\to\infty} \mathbb{E}^u\big(w_i'\big)\Big)$$

when $\Gamma \ni w_i \to w \leftarrow w_i' \in \Gamma$ as $i \to \infty$ and $w \notin \Gamma$. Deducing $\hat{\lambda}$-exponential dichotomy from semi $\hat{\lambda}$-exponential dichotomy, there is a similar difficulty. Therefore, our semi-hyperbolicity condition is weaker than that assumed in [5, Theorem C] and [12, Theorem B].

We, however, show in this paper that the semi-hyperbolicity condition implies hyperbolicity. Precisely, we prove the following result.

Theorem 1. *Let (θ, Θ) be semi $\hat{\lambda}$-exponentially dichotomous. If* $\mathrm{Index}(\hat{\lambda}; w)$ *is constant for a.e. $w \in W$, then (θ, Θ) is almost $\hat{\lambda}$-exponentially dichotomous on W; that is to say, there exists a continuous splitting of $\mathbb{R}^n$ into subspaces*

$$w \mapsto \mathbb{E}^u(w; \hat{\lambda}) \oplus \mathbb{E}^s(w; \hat{\lambda})$$

and constants $\varrho > 0$ and $C > 0$ such that for a.e. $w \in W$

$$\big\|\Theta(t+\bar{t}, w)\cdot x\big\| \leqslant Ce^{t(\hat{\lambda}-\varrho)}\big\|\Theta(\bar{t}, w)\cdot x\big\| \quad \forall x \in \mathbb{E}^s(w; \hat{\lambda})$$

for all $\bar{t} \in \mathbb{T}$ and for $t \in \mathbb{T}^+$, and

$$\big\|\Theta(t+\bar{t}, w)\cdot y\big\| \leqslant Ce^{t(\hat{\lambda}+\varrho)}\big\|\Theta(\bar{t}, w)\cdot y\big\| \quad \forall y \in \mathbb{E}^u(w; \hat{\lambda})$$

for all $\bar{t} \in \mathbb{T}$ and for $t \in \mathbb{T}^-$.

Notes:

(1) *This result implies immediately that (θ, Θ) is hyperbolic on a θ-invariant closed subset which equals W* mod 0 *if $\hat{\lambda} = 0$.*
(2) *If (W, θ) is uniquely ergodic, then* $\mathrm{Index}(\hat{\lambda}; w)$ *is constant for a.e. w by the Birkhoff ergodic theorem.*
(3) *If (θ, Θ) is almost nonuniformly contracting (resp. expanding), then we easily have* $\mathrm{Index}(0; w) = n$ *(resp.* 0*) for a.e. w.*
(4) *If $n = 2$ and (θ, Θ) is almost nonuniformly hyperbolic, then* $\mathrm{Index}(0; w) = 1$ *for a.e. $w \in W$.*
(5) *If (W, θ) is minimal and (θ, Θ) is semi $\hat{\lambda}$-exponentially dichotomous,* $\mathrm{Index}(\hat{\lambda}; w)$ *is constant for a.e. w from Definition* 1.

As an immediate corollary of this theorem, we have

Theorem 2. *Let $f: M \to M$ be a volume-preserving C^1 diffeomorphism of a compact, smooth, boundaryless Riemannian manifold M^n, $n \geqslant 2$. Suppose that*

$$\mathbb{D}: w \mapsto E(w) \quad \text{a.e. } w \in M$$

is a Df-invariant distribution, which is such that there exist constants $\lambda > 0$ and $c > 0$ with

$$\left\| D_w f^\ell \cdot \vec{v} \right\| \leqslant c\lambda^\ell \|\vec{v}\| \quad \forall \vec{v} \in E(w) \text{ and } \ell \geqslant 1$$

and

$$\limsup_{\ell \to +\infty} \frac{1}{\ell} \ln \left\| D_w f^\ell \cdot \vec{v} \right\| > \ln \lambda \quad \forall \vec{v} \in T_w M \setminus E(w)$$

for a.e. $w \in M$. Then, the following two statements hold:

(1) *If $\dim E(w)$ is constant for a.e. $w \in M$, then f is partially hyperbolic on M.*
(2) *If $\mathbb{D}: w \mapsto E(w)$ is continuous, then f is partially hyperbolic on M.*

Notice here that 'partially hyperbolic' means that there exist numbers $C > 0$ and $0 < \lambda < \mu$, and Df-invariant (continuous) splitting

$$T_w M = E(w) \oplus F(w) \quad \forall w \in M$$

such that

$$\left\| D_w f^\ell \cdot \vec{v} \right\| \leqslant C\lambda^\ell \|\vec{v}\| \quad \forall \vec{v} \in E(w)$$

and

$$\left\| D_w f^\ell \cdot \vec{v} \right\| \geqslant C^{-1}\mu^\ell \|\vec{v}\| \quad \forall \vec{v} \in F(w)$$

for $\ell \geqslant 1$; see [39, Definition 2.2].

Outline of proof of Theorem 1. We will carry the proof of this theorem assuming $\mathbb{T} = \mathbb{R}$ and Θ is almost uniformly $\hat{\lambda}$-contracting along the $\hat{\lambda}$-stable directions $\mathbb{E}^s(w; \hat{\lambda})$.

Given $w \in W$, we consider the family of linear isomorphisms

$$\big(\Theta(t, w)\big)_{t \in \mathbb{R}}: \mathbb{R}^n \to \mathbb{R}^n; \quad x \mapsto \Theta(t, w) \cdot x$$

and take an orthonormal basis of $\mathbb{R}^n$, say γ_w satisfying $\mathbb{E}(\mathrm{col}_1\gamma, \dots, \mathrm{col}_{\Bbbk}\gamma) = \mathbb{E}^s(w; \hat{\lambda})$, where $\Bbbk = \mathrm{Index}(\hat{\lambda}; w)$. Liao frame skew-product flow to be introduced in Section 2, naturally gives rise to a moving frame $(\gamma_w(t))_{t \in \mathbb{R}}$ which is a family of orthonormal bases of $\mathbb{R}^n$. Under this family of new moving coordinate systems, from $(\Theta(t, w))_{t \in \mathbb{R}}$ we obtain a new family of linear isomorphisms

$$\big(\varphi(t, \gamma_w)\big)_{t \in \mathbb{R}}: \mathbb{R}^n \to \mathbb{R}^n; \quad z \mapsto \varphi(t, \gamma_w) \cdot z,$$

namely,

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\Theta(t,w)\cdot} & \mathbb{R}^n \\ \uparrow{\scriptstyle \gamma_w\cdot} & & \uparrow{\scriptstyle \gamma_w(t)\cdot} \\ \mathbb{R}^n & \xrightarrow{\varphi(t,\gamma_w)\cdot} & \mathbb{R}^n \end{array} \qquad \forall t \in \mathbb{R}.$$

It turns out that $(\varphi(t,\gamma_w))_{t\in\mathbb{R}}$ is the fundamental matrix solution of the so-called Liao standard linear system to be introduced in Section 4

$$\frac{dz}{dt} = R_{\gamma_w}(t)z, \quad \text{where } R_{\gamma_w}(t) = \begin{bmatrix} \omega_1(\gamma_w(t)) & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_n(\gamma_w(t)) \end{bmatrix}.$$

Here $\omega_i(\cdot)$ are just the so-called Liao qualitative functions of the smooth linear skew-product system (θ,Θ), see (2.10) below.

The uniformly semi $\hat{\lambda}$-contracting condition at w enables us to find two constants $\eta > 0$ and $d > 0$ such that

$$\frac{1}{T}\int_0^T \omega_i\big(\gamma_w(t+t')\big)\,dt \leqslant \hat{\lambda} - \eta \quad (1 \leqslant i \leqslant \Bbbk) \tag{1.9a}$$

and

$$\frac{1}{T}\int_0^T \omega_j\big(\gamma_w(t+t')\big)\,dt \geqslant \hat{\lambda} + \eta \quad (\Bbbk < j \leqslant n) \tag{1.9b}$$

for any $t' \in \mathbb{R}$ and for any $T \geqslant d$. To prove this, we first use Liao's spectrum theorem [31,17] which provides us with integral expression of the Lyapunov exponents of (θ,Θ), and then use the semi-uniform ergodic theorem due to Sturman and Stark [41] which allows one to derive uniform estimates from nonuniform ones.

Furthermore, (1.9a) and (1.9b) guarantee that there exist two constants $\eta' > 0$ and $d' > 0$ and a unique splitting

$$\mathbb{R}^n = \mathbb{E}_-(\hat{\lambda}) \oplus \mathbb{E}_+(\hat{\lambda}), \quad \dim \mathbb{E}_-(\hat{\lambda}) = \Bbbk$$

such that

$$\big\|\varphi(s+t,\gamma_w)\cdot z\big\| \leqslant \big\|\varphi(s,\gamma_w)\cdot z\big\| \exp\big(t(\hat{\lambda}-\eta')\big) \quad \forall z \in \mathbb{E}_-(\hat{\lambda}) \tag{1.10a}$$

and

$$\big\|\varphi(s+t,\gamma_w)\cdot z\big\| \geqslant \big\|\varphi(s,\gamma_w)\cdot z\big\| \exp\big(t(\hat{\lambda}+\eta')\big) \quad \forall z \in \mathbb{E}_+(\hat{\lambda}) \tag{1.10b}$$

for any $s \in \mathbb{R}$ and for any $t \geqslant d'$. To prove this, we apply Liao's standard system theory [32, Chapter 2].

It is easy to see that (1.10a) and (1.10b) are almost what Theorem 1 requires.

By using the notations introduced in [16,15], we can obtain easily the proof of the discrete version of Theorem 1. □

1.2. Density of hyperbolic $GL_+(2,\mathbb{R})$-cocycles

In what follows, we consider discrete linear cocycles. Let $\theta : W \to W$ be a homeomorphism of the compact metrizable space W. Naturally it induces the discrete flow

$$\theta : \mathbb{Z} \times W \to W; \quad (t, w) \mapsto t.w.$$

Let $C^0(W, GL(n,\mathbb{R}))$ be the space of all continuous maps $A : W \to GL(n,\mathbb{R})$ of W into the space of invertible, $n \times n$, real matrices, endowed with the C^0-topology defined by the metric function

$$\operatorname{dist}(A, B) = \|A - B\| + \left\|A^{-1} - B^{-1}\right\|$$

for all $A, B \in C^0(W, GL(n,\mathbb{R}))$, where

$$\|M\| = \sup_{w \in W} \left(\sum_{i,j=1}^{n} M_{ij}(w)^2 \right)^{1/2} \quad \forall M \in C^0\bigl(W, GL(n,\mathbb{R})\bigr).$$

Then, to any $A \in C^0(W, GL(n,\mathbb{R}))$, based on (W, θ) there is a corresponding discrete skew-product system

$$(\theta, A) : \mathbb{Z} \times W \times \mathbb{R}^n \to W \times \mathbb{R}^n; \quad (t, w, x) \mapsto \bigl(t.w, A(t, w) \cdot x\bigr) \tag{1.11}$$

where $A(t, w) : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$A(t, w) = \begin{cases} A((t-1).w) \circ \cdots \circ A(w) & \text{if } t > 0, \\ \mathrm{Id} & \text{if } t = 0, \\ A^{-1}(t.w) \circ \cdots \circ A^{-1}(-1.w) & \text{if } t < 0. \end{cases} \tag{1.12}$$

Write

$$GL_+(n,\mathbb{R}) = \bigl\{M \in GL(n,\mathbb{R}) \mid M_{i,j} \geqslant 0\ 1 \leqslant i, j \leqslant n\bigr\}$$

and

$$SL(n,\mathbb{R}) = \bigl\{M \in GL(n,\mathbb{R}) \mid |\det M| = 1\bigr\}.$$

As a consequence of Theorem 1, we have the following hyperbolic result.

Theorem 3. *Let $A : W \to GL(n,\mathbb{R})$ be a positive continuous random matrix over (W, θ). If (θ, A) is almost nonuniformly hyperbolic with* $\operatorname{Codim} \mathbb{E}^s(w) = 1$ *a.e., then (θ, A) is almost uniformly hyperbolic on W.*

Here 'A is positive' means $A_{ij}(w) > 0$ for any $w \in W$ and $1 \leqslant i, j \leqslant n$.

Outline of proof of Theorem 3. Motivating by the Perron spectral theorem, we prove that there exists a random eigenvalue $\rho_A : W \to \mathbb{R}_+ - \{0\}$ and a corresponding random positive eigenvector $\xi_A : W \to \mathrm{Int}(\Delta) = \{u \in \mathbb{R}^n \mid \sum u_i = 1,\ u_i > 0\}$ (called positive core later), such that

$$A(w) \cdot \xi_A(w) = \rho_A(w)\xi_A(1.w) \quad \forall w \in W$$

and

$$\int_W \log \rho_A \, d\mu = \lambda_{\max}(A, \mu) \quad \forall \mu \in \mathcal{M}_{\mathrm{erg}}(W, \theta).$$

$\lambda_{\max}(A, \mu)$ is just the maximal Lyapunov exponent of $(\theta, \mu; A)$. As $\lambda_{\max}(A, \mu) > 0$ for any $\mu \in \mathcal{M}_{\mathrm{erg}}(W, \theta)$, it follows from the semi-uniform ergodic theorem (Lemma 3.1 below) that there are some $\eta > 0$ and $T_0 > 0$ such that

$$\left\| A(t + \bar{t}, w) \cdot \xi_A(w) \right\| \geqslant \left\| A(\bar{t}, w) \cdot \xi_A(w) \right\| \exp(t\eta)$$

for all $\bar{t} \in \mathbb{R}$, $t \geqslant T_0$ and for any $w \in W$. Thus, (θ, A) is semi-hyperbolic with uniformly expanding direction $\mathbb{E}^u(w) = \mathbb{E}(\xi_A(w))$. Then, Theorem 3 follows from Theorem 1. $\square$

Moreover, in the special case $n = 2$ we obtain the following open-and-dense result:

Theorem 4. *Let (W, θ) be uniquely ergodic with ν being the unique θ-invariant Borel probability measure on W. Then the following two statements hold*:

(1) *The set of $GL_+(2, \mathbb{R})$-valued cocycles based on (W, θ), which are either uniformly hyperbolic or uniformly expanding or uniformly contracting on the support of ν, is open and dense in $C^0(W, GL_+(2, \mathbb{R}))$.*
Note: *If we let $C^0_+(W, GL(2, \mathbb{R}))$ be the space of all eventually $GL_+(2, \mathbb{R})$-valued cocycles, then similarly the open-and-dense result still holds in this setting.*
(2) *If we further assume (W, θ) is equicontinuous, i.e., $(\theta^t : w \mapsto t.w)_{t=1}^{\infty}$ is equicontinuous, then the set of $GL(2, \mathbb{R})$-valued cocycles based on (W, θ), which are either uniformly hyperbolic or uniformly expanding or uniformly contracting on the support of ν, is open and dense in $C^0(W, GL(2, \mathbb{R}))$.*

Notice here that the interesting rotations $R_\alpha : \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$; $x \mapsto x + \alpha$ for any frequencies $\alpha \in \mathbb{R}^n$ are equicontinuous.

1.3. Density of $SL(2, \mathbb{R})$-cocycles having positive exponents

Finally, we pay our attention to the very interesting $SL(2, \mathbb{R})$-valued cocycles. Recall that for any $A \in C^0(W, GL(n, \mathbb{R}))$, if

$$\lim_{t \to \infty} \frac{1}{t} \log \left\| A(t, w) \right\| = \lambda(A) \tag{1.13}$$

holds for every $w \in W$ and uniformly on W, we call A to be *uniform* based on (W, θ); see [44]. It is easy to see that 'uniform' is not equivalent to 'uniformly hyperbolic.' However, the latter implies the former in the setting of $SL(2, \mathbb{R})$. Thus, Bochi's genericity theorem, [7, Theorem C], which is generalized to high dimension in [8], implies that uniform $SL(2, \mathbb{R})$-valued cocycle is generic over a fixed uniquely ergodic compact system. It is natural to ask

Question 2. Whether uniformly (resp. nonuniformly) hyperbolic $SL(2, \mathbb{R})$-valued cocycle is generic?

This question is not only of pure mathematical interest but is relevant in applications like, for example, the quantum mechanics of electrons in quasi-crystalline media.

In the sense of L^∞-topology, it was proved by Cong in [14] that a generic bounded $SL(2, \mathbb{R})$-cocycle is uniformly hyperbolic. Recently, Avila [3] proved that based on an irrational rotation of the d-dimensional torus, any analytic $SL(2, \mathbb{R})$-cocycle can be analytically perturbed so that the Lyapunov exponent becomes positive. In the present paper, in the sense of C^0-topology we obtain the density of $SL(2, \mathbb{R})$-cocycles with positive Lyapunov exponent based on a minimal subshift satisfying the Boshernitzan condition (see Definition 3 in Section 7 below).

Theorem 5. *Let (W, σ) be a minimal subshift that satisfies the Boshernitzan condition. Then, the $SL(2, \mathbb{R})$-valued cocycle, which has positive Lyapunov exponent based on σ, forms a dense subset in the space $C^0(W, SL(2, \mathbb{R}))$.*

Boshernitzan condition holds for a large number of subshifts [21]; for example, for all linearly repetitive subshifts, for all Sturmian subshifts, and for almost all Arnoux–Rauzy subshifts.

The density of positive exponents in $SL(2, \mathbb{R})$-cocycles based on a continuous-time flow $\theta : \mathbb{R} \times W \to W$ preserving a probability measure ν with support $\operatorname{supp}(\nu) = W$, was proved by Nerurkar in [36]. We relax the restriction $\operatorname{supp}(\nu) = W$ by applying a gluing lemma (Theorem A.1 below) proved in Appendix A. See Propositions A.2 and A.4 below.

1.4. Outlines

This paper is organized as follows. Sections 1–5 are devoted to the proof of Theorem 1. In Section 2, we introduce the necessary notion of Liao frame skew-product flow in Liao theory and prove some technical lemmas (Lemmas 2.1, 2.2 and 2.4 below). In addition, we recall Liao's spectrum theorem (Lemma 2.3 below) in this section. In Section 3, we prove a semi-uniform ergodic theorem (Lemma 3.2 below), which is one of our main tools. In Section 4, we first recall the Liao standard linear system (Definition 2 below) and then give a sufficient condition of λ-exponential dichotomy for the Liao standard linear systems (Lemma 4.2 below). Combining Lemma 4.2 with the geometric explanation of Liao standard linear system (Lemma 4.1 below), we finally finish the proofs of Theorem 1 and then Theorem 2 in Section 5.

We prove Theorem 3 in Section 6. Theorem 4 is also proved in this section based on Theorem 3 and using a genericity theorem of uniformity borrowed from [23]. In fact, we will consider linear cocycles over a compact system (W, θ) which need not be invertible and we will generalize a uniformity theorem of Walters (Theorem 6.14 below) by using natural extension.

We prove Theorem 5 in Section 7 by considering locally constant $SL(2, \mathbb{R})$-valued cocycles inspired by [20].

Appendix A is devoted to the density of nonuniform hyperbolicity in the $\mathbb{T}=\mathbb{R}$ setting. The gluing lemma (Theorem A.1 below) is the main result of this section.

2. Liao frame skew-product flow

In this section, we shall prove several technical lemmas. We assume throughout this section that $\mathbb{T}=\mathbb{R}$, and the skew-product flow (θ,Θ) and the constant $\hat\lambda$ are both given as in Theorem 1 stated in the Introduction.

2.1. Regular point set Γ

Let Γ be the Borel subset consisting of all Oseledets (θ,Θ)-regular points w in W. Let

$$\mathbb{D}_{\hat\lambda}: \Gamma \ni w \mapsto \mathbb{E}^u(w;\hat\lambda) \oplus \mathbb{E}^s(w;\hat\lambda) \tag{2.1}$$

be the Oseledets measurable (θ,Θ)-invariant splitting of $\mathbb{R}^n$ defined in the manner as in (1.4b) in Section 1.

From now on, there is no loss of generality in assuming that Γ is θ-invariant and there are constants $\varrho'>0$ and $C'>0$ such that

$$\big\|\Theta(t+\bar t, w)\cdot x\big\| \leqslant C' e^{t(\hat\lambda-\varrho')}\big\|\Theta(\bar t, w)\cdot x\big\| \quad \forall x\in\mathbb{E}^s(w;\hat\lambda) \tag{2.2a}$$

and

$$\Bbbk = \operatorname{Index}(\hat\lambda; w) \geqslant 1 \tag{2.2b}$$

for all $\bar t\in\mathbb{R}$, $t\geqslant 0$ and for any $w\in\Gamma$, because Γ is of total measure 1 and we need only to consider the skew-product flow

$$\big(\theta^{-1},\Theta^{-1}\big):\mathbb{R}\times W\times\mathbb{R}^n\to W\times\mathbb{R}^n;\quad (t,w,x)\mapsto\big(-t.w,\Theta(-t,w)\cdot x\big)$$

in the case where (1.6a) is true.

First, we have the following simple result.

Lemma 2.1. *Under the standard topology,*

$$\mathbb{D}_{\hat\lambda}: w\mapsto\mathbb{E}^s(w;\hat\lambda)$$

is continuous with respect to w in Γ; namely, $\mathbb{E}^s(w_i;\hat\lambda)\to\mathbb{E}^s(w;\hat\lambda)$ as $w_i\to w$ in Γ.

Proof. From (2.2) and the uniqueness of the Oseledets splitting (2.1) by

$$\mathbb{E}^s(w;\hat\lambda) = \left\{0, 0\neq x\in\mathbb{R}^n \;\Big|\; \lim_{t\to\pm\infty}\frac{1}{t}\log\big\|\Theta(t,w)\cdot x\big\| < \hat\lambda\right\} \tag{2.3a}$$

and

$$\mathbb{E}^u(w;\hat\lambda) = \left\{0, 0\neq x\in\mathbb{R}^n \;\Big|\; \lim_{t\to\pm\infty}\frac{1}{t}\log\big\|\Theta(t,w)\cdot x\big\| > \hat\lambda\right\} \tag{2.3b}$$

the statement follows immediately. $\square$

Important is the following simple property.

Lemma 2.2. *Γ is almost closed in W; that is to say, $\nu(\overline{\Gamma} - \Gamma) = 0$ for all $\nu \in \mathcal{M}_{\mathrm{erg}}(W, \theta)$.*

Proof. The statement comes immediately from $\nu\Gamma = 1$ for all $\nu \in \mathcal{M}_{\mathrm{erg}}(W, \theta)$. $\square$

2.2. Liao frame skew-product flow

Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$ with the usual inner product $\langle\cdot,\cdot\rangle$. As usual in Liao theory, we write

$$\mathscr{U}_n = \bigl\{\vec{\gamma} = (v_1, \dots, v_n) \mid \{v_1, \dots, v_n\} \text{ is a basis of } \mathbb{R}^n\bigr\} \tag{2.4a}$$

and

$$\mathscr{F}_n = \bigl\{\vec{\gamma} = (v_1, \dots, v_n) \mid \{v_1, \dots, v_n\} \text{ is an orthogonal basis of } \mathbb{R}^n\bigr\} \tag{2.4b}$$

and

$$\mathscr{F}_n^\sharp = \bigl\{\vec{\gamma} = (v_1, \dots, v_n) \mid \{v_1, \dots, v_n\} \text{ is an orthonormal basis of } \mathbb{R}^n\bigr\}. \tag{2.4c}$$

Notice here that $\mathscr{F}_n^\sharp$ and then $W \times \mathscr{F}_n^\sharp$ are both compact under the standard topologies. In what follows, we put

$$\mathrm{col}_k : \mathscr{U}_n \to \mathbb{R}^n; \quad (v_1, \dots, v_n) \mapsto v_k,\ 1 \leqslant k \leqslant n. \tag{2.5}$$

Let

$$\Theta_n(t, w) : \mathscr{U}_n \to \mathscr{U}_n; \quad \vec{\gamma} \mapsto \bigl(\Theta(t, w) \cdot \mathrm{col}_1\, \vec{\gamma}, \dots, \Theta(t, w) \cdot \mathrm{col}_n\, \vec{\gamma}\bigr) \tag{2.6a}$$

for all $(t, w) \in \mathbb{R} \times W$, we then have the following skew-product flow

$$(\theta, \Theta_n) : \mathbb{R} \times W \times \mathscr{U}_n \to W \times \mathscr{U}_n; \quad (t, w, \vec{\gamma}) \mapsto \bigl(t.w, \Theta_n(t, w) \cdot \vec{\gamma}\bigr). \tag{2.6b}$$

Let

$$\mathrm{Ort} : \mathscr{U}_n \to \mathscr{F}_n \quad \text{and} \quad \mathrm{Ort}^\sharp : \mathscr{U}_n \to \mathscr{F}_n^\sharp \tag{2.7}$$

denote, respectively, the usual Gram–Schmidt orthogonalization and orthonormalization transformations. If we put

$$\widehat{\Theta}_n(t, w) : \mathscr{F}_n \to \mathscr{F}_n; \quad \vec{\gamma} \mapsto \mathrm{Ort} \circ \Theta_n(t, w) \cdot \vec{\gamma} \tag{2.8a}$$

for all $(t, w) \in \mathbb{R} \times W$, then we obtain the following skew-product flow

$$(\theta, \widehat{\Theta}_n) : \mathbb{R} \times W \times \mathscr{F}_n \to W \times \mathscr{F}_n; \quad (t, w, \vec{\gamma}) \mapsto \bigl(t.w, \widehat{\Theta}_n(t, w) \cdot \vec{\gamma}\bigr). \tag{2.8b}$$

Similarly, we obtain the following so-called *Liao frame skew-product flow*:

$$\big(\theta, \widehat{\Theta}_n^\sharp\big): \mathbb{R} \times W \times \mathscr{F}_n^\sharp \to W \times \mathscr{F}_n^\sharp; \quad (t, w, \vec{\gamma}) \mapsto \big(t.w, \widehat{\Theta}_n^\sharp(t, w) \cdot \vec{\gamma}\big) \tag{2.9a}$$

where

$$\widehat{\Theta}_n^\sharp(t, w): \vec{\gamma} \mapsto \mathrm{Ort}^\sharp \circ \Theta_n(t, w) \cdot \vec{\gamma}. \tag{2.9b}$$

Next, we define the continuous functions

$$\omega_k : W \times \mathscr{F}_n^\sharp \to \mathbb{R} \quad (k = 1, \dots, n) \tag{2.10}$$

in the way

$$\omega_k(w, \vec{\gamma}) = \frac{d}{dt}\bigg|_{t=0} \big\|\mathrm{col}_k \circ \widehat{\Theta}_n(t, w) \cdot \vec{\gamma}\big\| \quad \forall (w, \vec{\gamma}) \in W \times \mathscr{F}_n^\sharp.$$

Since Θ is smooth, ω_k is well defined. These functions are called the *Liao qualitative functions* of (θ, Θ) on $W \times \mathscr{F}_n^\sharp$ introduced by Liao in [29] (or see [31,17]). It is easy to see [29,31,17] that

$$\log\big\|\mathrm{col}_k \circ \widehat{\Theta}_n(T, w) \cdot \vec{\gamma}\big\| = \int_0^T \omega_k\big(\big(\theta, \widehat{\Theta}_n^\sharp\big)(t, w, \vec{\gamma})\big)\, dt \quad \forall T \in \mathbb{R} \tag{2.11}$$

for any $(w, \vec{\gamma}) \in W \times \mathscr{F}_n^\sharp$.

Since the skew-product system $(\theta, \widehat{\Theta}_n^\sharp)$ is compact, we can apply the ergodic theory. For example, for any $\nu \in \mathcal{M}_{\mathrm{erg}}(W, \theta)$ there is at least one $\mu \in \mathcal{M}_{\mathrm{erg}}(W \times \mathscr{F}_n^\sharp, (\theta, \widehat{\Theta}_n^\sharp))$ which covers ν; i.e., $\pi_*\mu = \nu$, where

$$\pi : W \times \mathscr{F}_n^\sharp \ni (w, \vec{\gamma}) \mapsto w \in W \tag{2.12}$$

is the natural bundle projection.

By $\mathsf{Sp}(\theta, \nu; \Theta)$ we denote the set of all Lyapunov exponents of (θ, Θ), *counting with multiplicity*, associated to ν in $\mathcal{M}_{\mathrm{erg}}(W, \theta)$. Important is the following *Liao spectrum theorem*:

Lemma 2.3. *(See [31,17].) Given any $\nu \in \mathcal{M}_{\mathrm{erg}}(W, \theta)$. If $\mu \in \mathcal{M}_{\mathrm{erg}}(W \times \mathscr{F}_n^\sharp, (\theta, \widehat{\Theta}_n^\sharp))$ satisfies $\pi_*\mu = \nu$, then*

$$\mathsf{Sp}(\theta, \nu; \Theta) = \left\{ \int_{W \times \mathscr{F}_n^\sharp} \omega_k(w, \vec{\gamma})\, d\mu(w, \vec{\gamma}) \;\middle|\; k = 1, \dots, n \right\}.$$

This lemma is one of our crucial tools, which together with (2.11) enables us to use the semi-uniform ergodic theorem established below.

2.3. Subsystems of Liao frame skew-product flow

Next, we shall pay our attention to the compact subsystems

$$(\theta, \Theta) : \mathbb{R} \times \overline{\Gamma} \times \mathbb{R}^n \to \overline{\Gamma} \times \mathbb{R}^n \tag{2.13a}$$

and

$$\big(\theta, \widehat{\Theta}_n^{\sharp}\big) : \mathbb{R} \times \overline{\Gamma} \times \mathscr{F}_n^{\sharp} \to \overline{\Gamma} \times \mathscr{F}_n^{\sharp}. \tag{2.13b}$$

Given ℓ vectors $v_1, \ldots, v_\ell$ in $\mathbb{R}^n$, we denote by $\mathbb{E}(v_1, \ldots, v_\ell)$ the linear subspace of $\mathbb{R}^n$ spanned by $\{v_1, \ldots, v_\ell\}$. Let $\Bbbk$ be the $\hat{\lambda}$-index of (θ, Θ) as in (2.2b). For any regular point $w \in \Gamma$, we set

$$\mathcal{F}^{\sharp}(w) = \big\{(w, \vec{\gamma}) \bigm| \vec{\gamma} \in \mathscr{F}_n^{\sharp} \text{ such that } \mathbb{E}(\operatorname{col}_1 \vec{\gamma}, \ldots, \operatorname{col}_{\Bbbk} \vec{\gamma}) = \mathbb{E}^s(w; \hat{\lambda})\big\}. \tag{2.14}$$

Then,

$$\mathcal{F}^{\sharp}(\Gamma) = \bigcup_{w \in \Gamma} \mathcal{F}^{\sharp}(w) \tag{2.15}$$

is a $(\theta, \widehat{\Theta}_n^{\sharp})$-invariant subbundle of $W \times \mathscr{F}_n^{\sharp}$.

In what follows, we denote by $\overline{\mathcal{F}^{\sharp}(\Gamma)}$ the closure of $\mathcal{F}^{\sharp}(\Gamma)$ in $W \times \mathscr{F}_n^{\sharp}$.

The following lemma is important for the proof of Theorem 1.

Lemma 2.4. *Let $\pi : W \times \mathscr{F}_n^{\sharp} \ni (w, \vec{\gamma}) \mapsto w$ be the natural bundle projection. Then, we have $\pi(\overline{\mathcal{F}^{\sharp}(\Gamma)} - \mathcal{F}^{\sharp}(\Gamma)) = \overline{\Gamma} - \Gamma$.*

Proof. Given any $w \in \overline{\Gamma} - \Gamma$. There is a sequence (w_i) in Γ such that $w_i \to w$ as $i \to \infty$. Letting $(w_i, \vec{\gamma}_i) \in \mathcal{F}^{\sharp}(\Gamma)$ satisfying $\vec{\gamma}_i \to \vec{\gamma}$ from the compactness of $\mathscr{F}_n^{\sharp}$, we easily have $(w_i, \vec{\gamma}_i) \to (w, \vec{\gamma}) \in \overline{\mathcal{F}^{\sharp}(\Gamma)} - \mathcal{F}^{\sharp}(\Gamma)$ and $\pi(w, \vec{\gamma}) = w$. This shows

$$\pi\big(\overline{\mathcal{F}^{\sharp}(\Gamma)} - \mathcal{F}^{\sharp}(\Gamma)\big) \supseteq \overline{\Gamma} - \Gamma.$$

On the other hand, let $(w, \vec{\gamma}) \in \overline{\mathcal{F}^{\sharp}(\Gamma)} - \mathcal{F}^{\sharp}(\Gamma)$. We easily get $w \in \overline{\Gamma}$ and there is a sequence $(w_i, \vec{\gamma}_i)$ in $\mathcal{F}^{\sharp}(\Gamma)$ with $(w_i, \vec{\gamma}_i) \to (w, \vec{\gamma})$ as $i \to \infty$. If we assume $w \in \Gamma$, then

$$\mathbb{E}(\operatorname{col}_1 \vec{\gamma}_i, \ldots, \operatorname{col}_{\Bbbk} \vec{\gamma}_i) = \mathbb{E}^s(w_i; \hat{\lambda}) \to \mathbb{E}^s(w; \hat{\lambda}) = \mathbb{E}(\operatorname{col}_1 \vec{\gamma}, \ldots, \operatorname{col}_{\Bbbk} \vec{\gamma}) \quad \text{as } i \to \infty$$

by Lemma 2.1, which implies $(w, \vec{\gamma}) \in \mathcal{F}^{\sharp}(\Gamma)$. This is a contradiction. Hence, $w \in \overline{\Gamma} - \Gamma$, as desired.

The proof is thus completed. □

3. Semi-uniform ergodic theorem

In order to prove our semi-uniform result (Lemma 3.2 below) which is a crucial tool, we need a general semi-uniform Birkhoff ergodic theorem which is a continuous-time version of one due to Herman [25] and Sturman and Stark [41].

Lemma 3.1. *Suppose that $\Upsilon : \mathbb{R} \times Y \to Y$ is a C^0-flow defined on a compact metrizable space Y and $\omega : Y \to \mathbb{R}$ a continuous function.*

(1) *If there exists a constant $a \in \mathbb{R}$ such that*

$$\int_Y \omega \, d\mu < a \quad \forall \mu \in \mathcal{M}_{\mathrm{erg}}(Y, \Upsilon),$$

then there exists some $\delta > 0$ such that

$$\int_Y \omega \, d\mu \leqslant a - \delta \quad \forall \mu \in \mathcal{M}_{\mathrm{erg}}(Y, \Upsilon)$$

and given $\varepsilon > 0$, there is $T_0 > 0$ such that for all $T \geqslant T_0$ we have

$$\frac{1}{T} \int_0^T \omega\big(\Upsilon(t+s, y)\big) \, dt < a - \delta + \varepsilon$$

for all $y \in Y$ and for any $-\infty < s < +\infty$ [41].

(2) *If there exists a constant $b \in \mathbb{R}$ such that*

$$\int_Y \omega \, d\mu > b \quad \forall \mu \in \mathcal{M}_{\mathrm{erg}}(Y, \Upsilon),$$

then, there exists some $\delta > 0$ such that

$$\int_Y \omega \, d\mu \geqslant b + \delta \quad \forall \mu \in \mathcal{M}_{\mathrm{erg}}(Y, \Upsilon)$$

and given $\varepsilon > 0$, there is $T_0 > 0$ such that for all $T \geqslant T_0$ we have

$$\frac{1}{T} \int_0^T \omega\big(\Upsilon(t+s, y)\big) \, dt > b + \delta - \varepsilon$$

for all $y \in Y$ and for any $-\infty < s < +\infty$.

(3) *If there exists a constant* $c \in \mathbb{R}$ *such that*

$$\int_Y \omega\, d\mu = c \quad \forall \mu \in \mathcal{M}_{\mathrm{erg}}(Y, \Upsilon),$$

then, as T *tends to* ∞ *the time average*

$$\frac{1}{T}\int_0^T \omega\big(\Upsilon(t, y)\big)\, dt \to c$$

uniformly for all $y \in Y$ [25].

From now on, we let the skew-product system (θ, Θ) be given as in Theorem 1 and Γ be defined as in Section 2.1. Consider the skew-product flow

$$\big(\theta, \widehat{\Theta}_n^{\sharp}\big) : \mathbb{R} \times \overline{\mathcal{F}^{\sharp}(\Gamma)} \to \overline{\mathcal{F}^{\sharp}(\Gamma)}; \quad (t, w, \vec{\gamma}) \mapsto \big(t.w, \widehat{\Theta}_n^{\sharp}(t, w) \cdot \vec{\gamma}\big) \tag{3.1}$$

where $\theta : \mathbb{R} \times \overline{\Gamma} \to \overline{\Gamma}$ is the subsystem by restricting the base system (W, θ) on $\overline{\Gamma}$, and where $\widehat{\Theta}_n^{\sharp}(t, w) \cdot \vec{\gamma}$ is as in (2.9b) to be restricted on $\overline{\mathcal{F}^{\sharp}(\Gamma)}$. Let $\omega_k(w, \vec{\gamma})$ be the Liao qualitative functions as in (2.10) and $\Bbbk$ as in (2.2b).

The following semi-uniform ergodic theorem is one of our main tools.

Lemma 3.2. *Let* $(\theta, \widehat{\Theta}_n^{\sharp})$ *be as in* (3.1). *Then, for any* $\mu \in \mathcal{M}_{\mathrm{erg}}(\overline{\mathcal{F}^{\sharp}(\Gamma)}, (\theta, \widehat{\Theta}_n^{\sharp}))$ *we have*

$$\int_{\overline{\mathcal{F}^{\sharp}(\Gamma)}} \omega_i\, d\mu < \hat{\lambda} \quad \textit{for } 1 \leqslant i \leqslant \Bbbk$$

and

$$\int_{\overline{\mathcal{F}^{\sharp}(\Gamma)}} \omega_i\, d\mu > \hat{\lambda} \quad \textit{for } \Bbbk < i \leqslant n \textit{ if } \Bbbk < n.$$

Consequently, there exist two constants $\eta > 0$ *and* $T_0 > 0$ *such that*

$$\frac{1}{T}\int_0^T \omega_i\big(\big(\theta, \widehat{\Theta}_n^{\sharp}\big)(t + t', w, \vec{\gamma})\big)\, dt \leqslant \hat{\lambda} - \eta \quad (1 \leqslant i \leqslant \Bbbk)$$

and

$$\frac{1}{T}\int_0^T \omega_i\big(\big(\theta, \widehat{\Theta}_n^{\sharp}\big)(t + t', w, \vec{\gamma})\big)\, dt \geqslant \hat{\lambda} + \eta \quad (\Bbbk < i \leqslant n \textit{ if } \Bbbk < n)$$

for all $(w, \vec{\gamma}) \in \overline{\mathcal{F}^{\sharp}(\Gamma)}$ *and for any* $t' \in \mathbb{R}$, $T \geqslant T_0$.

Proof. According to the statements (1) and (2) of Lemma 3.1, we need only to prove the first part of Lemma 3.2 by choosing $\eta = \delta - \varepsilon$.

Given any $\mu \in \mathcal{M}_{\mathrm{erg}}(\overline{\mathcal{F}^\sharp(\Gamma)}, (\theta, \widehat{\Theta}_n^\sharp))$. Let

$$\nu = \pi_* \mu.$$

It is clear that ν lies in $\mathcal{M}_{\mathrm{erg}}(\overline{\Gamma}, \theta)$, since $\pi(\overline{\mathcal{F}^\sharp(\Gamma)}) = \overline{\Gamma}$ and

$$\begin{tikzcd}
\mathbb{R} \times \overline{\mathcal{F}^\sharp(\Gamma)} \arrow[r, "(\theta, \widehat{\Theta}_n^\sharp)"] \arrow[d, "\mathrm{Id} \times \pi"'] & \overline{\mathcal{F}^\sharp(\Gamma)} \arrow[d, "\pi"] \\
\mathbb{R} \times \overline{\Gamma} \arrow[r, "\theta"] & \overline{\Gamma}
\end{tikzcd}$$

is commutative.

Write

$$\mathsf{Sp}(\theta, \nu; \Theta) = \{\chi_1, \ldots, \chi_n\}$$

where

$$\chi_1 \leqslant \cdots \leqslant \chi_{\Bbbk} < \hat{\lambda} < \chi_{\Bbbk+1} \leqslant \cdots \leqslant \chi_n. \tag{3.2}$$

As $\nu(\overline{\Gamma} - \Gamma) = 0$ by Lemma 2.2, it follows from Lemma 2.4 that

$$\mu\big(\overline{\mathcal{F}^\sharp(\Gamma)} - \mathcal{F}^\sharp(\Gamma)\big) = 0. \tag{3.3}$$

Therefore, by Lemma 2.3 we obtain

$$\mathsf{Sp}(\theta, \nu; \Theta) = \left\{ \int\limits_{\overline{\mathcal{F}^\sharp(\Gamma)}} \omega_i \, d\mu \;\middle|\; i = 1, \ldots, n \right\} = \left\{ \int\limits_{\mathcal{F}^\sharp(\Gamma)} \omega_i \, d\mu \;\middle|\; i = 1, \ldots, n \right\} \tag{3.4}$$

and moreover, by (2.14) we have

$$\{\chi_1, \ldots, \chi_{\Bbbk}\} = \left\{ \int\limits_{\mathcal{F}^\sharp(\Gamma)} \omega_i \, d\mu \;\middle|\; i = 1, \ldots, \Bbbk \right\} \tag{3.5a}$$

and then

$$\{\chi_{\Bbbk+1}, \ldots, \chi_n\} = \left\{ \int\limits_{\mathcal{F}^\sharp(\Gamma)} \omega_i \, d\mu \;\middle|\; i = \Bbbk + 1, \ldots, n \right\} \tag{3.5b}$$

ignoring the order. (3.5a) and (3.5b) together with (3.2) imply the required statement.

Thus, we proved the result. $\square$

4. $\hat{\lambda}$-exponential dichotomy

In this section, we shall prove that for any $w \in \overline{\Gamma}$, the family of linear isomorphisms $(\Theta(t, w))_{t\in\mathbb{R}} : \mathbb{R}^n \to \mathbb{R}^n$ possesses a $\hat{\lambda}$-exponential dichotomy uniformly with respect to a.e. w in the situations of Theorem 1.

In what follows, under the standard basis $\vec{e} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$, we view z in $\mathbb{R}^n$ as a column vector $(z_1, \dots, z_n)^T$ and $\vec{\gamma} \in \mathscr{F}_n^\sharp$ as a row 'vector' or an n-by-n matrix with columns $\mathrm{col}_1\,\vec{\gamma}, \dots, \mathrm{col}_n\,\vec{\gamma}$, successively.

4.1. Liao standard linear systems

In order to apply Lemma 3.2 proved in Section 3, we need to use the notation of Liao standard linear systems associated to (θ, Θ) as in (1.1) considered in the Introduction. Let

$$\pi : W \times \mathbb{R}^n \to W; \quad (w, z) \mapsto w$$

be the natural bundle projection. Then, write the fiber $\pi^{-1}(w)$ as $\mathbb{R}^n(w)$ for any base $w \in W$.

For any orthonormal n-frame $(w, \vec{\gamma}) \in W \times \mathscr{F}_n^\sharp$ at the base $w \in W$, simply written as $\vec{\gamma}_w$, we define by linear extension the linear transformation

$$\mathcal{T}_{\vec{\gamma}} : \mathbb{R}^n \to \mathbb{R}^n(w) \tag{4.1}$$

in the way

$$\mathbf{e}_k \mapsto \mathrm{col}_k\,\vec{\gamma} \quad (1 \leqslant k \leqslant n)$$

where col_k is defined as in (2.5). Since $\vec{\gamma}$ is an orthonormal basis of $\mathbb{R}^n(w)$, $\mathcal{T}_{\vec{\gamma}}$ is an isomorphism such that

$$\mathcal{T}_{\vec{\gamma}}(\vec{e} \cdot z) = \vec{\gamma} \cdot z \quad \text{and} \quad \|\vec{e} \cdot z\| = \|\vec{\gamma} \cdot z\|$$

for any $z = (z_1, \dots, z_n)^T \in \mathbb{R}^n$.

For any given $(w, \vec{\gamma}) \in W \times \mathscr{F}_n^\sharp$, we now define

$$C_{\vec{\gamma}_w}(t) = \mathcal{T}^{-1}_{\widehat{\Theta}_n^\sharp(t,w)\cdot\vec{\gamma}} \circ \Theta(t, w) \circ \mathcal{T}_{\vec{\gamma}} \quad \forall t \in \mathbb{R}, \tag{4.2}$$

where $\widehat{\Theta}_n^\sharp(t, w) : \mathscr{F}_n^\sharp \to \mathscr{F}_n^\sharp$ is the Liao frame skew-product flow as in (2.9b) induced by (θ, Θ). Then the commutativity holds:

$$\begin{array}{ccc} \mathbb{R}^n(w) & \xrightarrow{\Theta(t,w)\cdot} & \mathbb{R}^n(t.w) \\ \mathcal{T}_{\vec{\gamma}} \Big\uparrow & & \Big\uparrow \mathcal{T}_{\widehat{\Theta}_n^\sharp(t,w)\cdot\vec{\gamma}} \\ \mathbb{R}^n & \xrightarrow{C_{\vec{\gamma}_w}(t)\cdot} & \mathbb{R}^n. \end{array} \tag{4.3}$$

We now see $C_{\vec{\gamma}_w}(t)$ as an $(n \times n)$-matrix. Clearly, $\frac{d}{dt}C_{\vec{\gamma}_w}(t)$ makes sense since (θ, Θ) is a smooth linear skew-product flow and we have

$$C_{\vec{\gamma}_w}(\tau + t) = C_{(\theta, \widehat{\Theta}_n^\sharp)(t,w,\vec{\gamma})}(\tau) \circ C_{\vec{\gamma}_w}(t) \quad \forall t, \tau \in \mathbb{R}. \tag{4.4}$$

Put

$$R(\vec{\gamma}_w) = \frac{d}{dt}\bigg|_{t=0} C_{\vec{\gamma}_w}(t) \quad \forall (w, \vec{\gamma}) \in W \times \mathscr{F}_n^\sharp \tag{4.5a}$$

and simply write

$$t.(w, \vec{\gamma}) = \big(\theta, \widehat{\Theta}_n^\sharp\big)(t, w, \vec{\gamma}) \quad \forall (t, w, \vec{\gamma}) \in \mathbb{R} \times W \times \mathscr{F}_n^\sharp. \tag{4.5b}$$

Definition 2. Based on (W, θ) as in (1.1), the linear equation

$$\frac{dz}{dt} = R\big(t.(w, \vec{\gamma})\big)z, \quad (t, z) \in \mathbb{R} \times \mathbb{R}^n, \tag{$R_{\vec{\gamma}_w}$}$$

for any $(w, \vec{\gamma}) \in W \times \mathscr{F}_n^\sharp$, is called the *Liao standard linear system* of (θ, Θ) under the moving base $(w, \vec{\gamma})$; see [31,17].

We will need the following

Lemma 4.1. *(See [17, Theorem 3.4, Proposition 3.5].) Let (θ, Θ) be as in* (1.1)*. Then Eq.* $(R_{\vec{\gamma}_w})$ *has the following properties*:

(1) *Uniform boundedness*:

$$\sup_{(w,\vec{\gamma}) \in W \times \mathscr{F}_n^\sharp} \bigg(\sum_{i,j} R_{ij}(\vec{\gamma}_w)^2\bigg)^{1/2} < \infty.$$

(2) *Upper-triangularity*: *For any $(w, \vec{\gamma}) \in W \times \mathscr{F}_n^\sharp$ and $t \in \mathbb{R}$, $R(t.(w, \vec{\gamma}))$ is upper-triangular with diagonal elements*

$$R_{kk}\big(t.(w, \vec{\gamma})\big) = \omega_k\big(t.(w, \vec{\gamma})\big) \quad (k = 1, \ldots, n)$$

where ω_k are the Liao qualitative functions as in (2.10).

(3) *Geometrical interpretation*: *For any $x \in \mathbb{R}^n$, $z(t, x)$ is the solution of Eq.* $(R_{\vec{\gamma}_w})$ *satisfying the initial condition $z(0) = x$ if and only if*

$$\Theta(t, w)(\vec{\gamma} \cdot x) = \big(\widehat{\Theta}_n^\sharp(t, w) \cdot \vec{\gamma}\big) \cdot z(t, x) \quad \forall t \in \mathbb{R}.$$

Particularly, $C_{\vec{\gamma}_w}(t)$ is the fundamental matrix solution of Eq. $(R_{\vec{\gamma}_w})$ *such that $C_{\vec{\gamma}_w}(0) = I_n$.*

4.2. An abstraction

In the following, we consider an abstract linear equation

$$\frac{dz}{dt} = B(t)z, \quad t \in \mathbb{R}, z = (z_1, \dots, z_n)^T \in \mathbb{R}^n, \tag{4.6}$$

where

$$B(t) = \begin{bmatrix} b_{11}(t) & b_{12}(t) & \cdots & b_{1n}(t) \\ 0 & b_{22}(t) & \cdots & b_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & b_{nn}(t) \end{bmatrix} \tag{4.7}$$

is a real n-by-n upper-triangular matrix, which is continuous in t such that

$$\|B\| = \sup_{t\in\mathbb{R}} \left\{ \sum_{1\leqslant i,j\leqslant n} b_{ij}(t)^2 \right\}^{1/2} \leqslant \eta < \infty. \tag{4.8}$$

Let $z = z(t,x)$ be the solution of (4.6) such that $z(0,x) = x$ for any $x \in \mathbb{R}^n$.

The following lemma gives a sufficient condition of $\hat{\lambda}$-exponential dichotomy, which is our another main tool. It is a generalization of [30, Lemma 3.7]. Although its proof is only a slight improvement of that of Liao's result, we shall prove it in detail for completeness, since it is difficult for many readers to read and find Liao's papers.

Lemma 4.2. *Let $\lambda \in \mathbb{R}$ and $\{1,\dots,n\} = \mathrm{I}_- \cup \mathrm{I}_+$ be a partition, and suppose that there are constants $\eta > 0$ and $d > 0$ such that*

$$\frac{1}{T}\int_0^T b_{kk}(s+t)\,dt \leqslant \lambda - \eta \quad \forall k \in \mathrm{I}_-, s \in \mathbb{R} \text{ and } T \geqslant d \tag{4.9a}$$

and

$$\frac{1}{T}\int_0^T b_{kk}(s+t)\,dt \geqslant \lambda + \eta \quad \forall k \in \mathrm{I}_+, s \in \mathbb{R} \text{ and } T \geqslant d. \tag{4.9b}$$

Then, there exists a unique direct decomposition of $\mathbb{R}^n$ into subspaces

$$\mathbb{R}^n = \mathbb{E}_-(\lambda) \oplus \mathbb{E}_+(\lambda)$$

such that there are constants $\eta' > 0$ and $d' > 0$ verifying that

$$\big\|z(s+T,x)\big\| \leqslant \big\|z(s,x)\big\| e^{(\lambda-\eta')T} \quad \forall (s,x) \in \mathbb{R}\times\mathbb{E}_-(\lambda) \text{ and } T \geqslant d' \tag{4.10a}$$

and

$$\|z(s+T,x)\| \geqslant \|z(s,x)\| e^{(\lambda+\eta')T} \quad \forall (s,x) \in \mathbb{R} \times \mathbb{E}_+(\lambda) \text{ and } T \geqslant d' \tag{4.10b}$$

and $\dim \mathbb{E}_-(\lambda) = \operatorname{Card} \mathrm{I}_-$ *and* $\dim \mathbb{E}_+(\lambda) = \operatorname{Card} \mathrm{I}_+$. *Moreover, η' and d' are both determined completely by λ, η, d and* $\daleth$. *In addition, if* $\mathrm{I}_- = \{1, \ldots, \operatorname{Card} \mathrm{I}_-\}$ *then* $\mathbb{E}_-(\lambda) = \mathbb{R}^{\operatorname{Card} \mathrm{I}_-} \times \{0_{n-\operatorname{Card} \mathrm{I}_-}\}$.

Proof. By the coordinate transformations

$$\tilde{z} = e^{-\lambda t} z \tag{4.11}$$

we obtain from (4.6) the following linear equation

$$\frac{d\tilde{z}}{dt} = \widetilde{B}(t)\tilde{z} \tag{4.12a}$$

where

$$\widetilde{B}(t) = \left[\tilde{b}_{ij}(t)\right]_{n\times n} = \operatorname{diag}(-\lambda, \ldots, -\lambda) + B(t) \quad \forall t \in \mathbb{R}. \tag{4.12b}$$

Then, we have

$$\frac{1}{T}\int_0^T \tilde{b}_{kk}(s+t)\,dt \leqslant -\eta \quad \forall k \in \mathrm{I}_-,\ s \in \mathbb{R} \text{ and } T \geqslant d \tag{4.13a}$$

and

$$\frac{1}{T}\int_0^T \tilde{b}_{kk}(s+t)\,dt \geqslant \eta \quad \forall k \in \mathrm{I}_+, s \in \mathbb{R} \text{ and } T \geqslant d. \tag{4.13b}$$

Write

$$\widetilde{B}(t) = \begin{bmatrix} \tilde{b}_{11}(t) & \tilde{b}(t) \\ 0_{(n-1)\times 1} & Y(t) \end{bmatrix} \quad \forall t \in \mathbb{R}. \tag{4.13c}$$

For any $x \in \mathbb{R}^n$, let $\tilde{z}(t,x)$ be the solution of (4.12a) such that $\tilde{z}(0,x) = x$.

We first prove the result for (4.12a) by induction on the order n; that is to say, we prove the claim that there exists a unique direct decomposition of $\mathbb{R}^n$ into subspaces

$$\mathbb{R}^n = \widetilde{\mathbb{E}}_- \oplus \widetilde{\mathbb{E}}_+$$

such that there are constants $\eta' > 0$ and $d' > 0$ such that

$$\|\tilde{z}(s+T,x)\| \leqslant \|\tilde{z}(s,x)\| e^{-\eta' T} \quad \forall (s,x) \in \mathbb{R} \times \widetilde{\mathbb{E}}_- \text{ and } T \geqslant d' \tag{4.14a}$$

and

$$\|\tilde{z}(s+T,x)\| \geqslant \|\tilde{z}(s,x)\| e^{\eta' T} \quad \forall (s,x) \in \mathbb{R} \times \widetilde{\mathbb{E}}_+ \text{ and } T \geqslant d' \tag{4.14b}$$

and $\dim \widetilde{\mathbb{E}}_- = \operatorname{Card} \mathrm{I}_-$ and $\dim \widetilde{\mathbb{E}}_+ = \operatorname{Card} \mathrm{I}_+$. Moreover, η' and d' are both determined completely by η, d and $\tilde{\eta}$, where $\tilde{\eta}$ is defined by the manner as in (4.8) for $\widetilde{B}(t)$.

As for $n = 1$ we have

$$\tilde{z}(s+T,x) = \tilde{z}(s,x) e^{\int_0^T \tilde{b}_{11}(s+t)\,dt},$$

the claim obviously holds if we let $\eta' = \eta$ and $d' = d$.

Therefore, in what follows we assume $n > 1$.

Let

$$\begin{aligned} &\mathrm{Pr}_1 : \mathbb{R}^n \to \mathbb{R}; \quad (x_1, \dots, x_n)^T \mapsto x_1, \\ &\mathrm{Pr}_{2\cdots n} : \mathbb{R}^n \to \mathbb{R}^{n-1}; \quad (x_1, \dots, x_n)^T \mapsto (x_2, \dots, x_n)^T. \end{aligned} \tag{4.15}$$

Then, for any $\tilde{z} \in \mathbb{R}^n$ we can write $\tilde{z} = (\mathrm{Pr}_1 \tilde{z}, \mathrm{Pr}_{2\cdots n} \tilde{z})^T$.

For any $x \in \mathbb{R}^n$ we easily get

$$\frac{d}{dt} \mathrm{Pr}_1 \tilde{z}(t,x) = \tilde{b}_{11}(t) \mathrm{Pr}_1 \tilde{z}(t,x) + \tilde{b}(t) \mathrm{Pr}_{2\cdots n} \tilde{z}(t,x) \tag{4.16a}$$

and

$$\frac{d}{dt} \mathrm{Pr}_{2\cdots n} \tilde{z}(t,x) = Y(t) \mathrm{Pr}_{2\cdots n} \tilde{z}(t,x). \tag{4.16b}$$

Hence, from (4.16a) it follows

$$\begin{aligned} \mathrm{Pr}_1 \tilde{z}(s+T,x) = {} & \mathrm{Pr}_1 \tilde{z}(s,x) e^{\int_0^T \tilde{b}_{11}(s+t)\,dt} \\ & + \int_0^T \tilde{b}(s+t) \mathrm{Pr}_{2\cdots n} \tilde{z}(s+t,x) e^{\int_t^T \tilde{b}_{11}(s+\tau)\,d\tau}\,dt. \end{aligned} \tag{4.17}$$

Let $y = y(t,x')$ be the solution of the equation

$$\frac{dy}{dt} = Y(t) y, \quad (t,y) \in \mathbb{R} \times \mathbb{R}^{n-1}, \tag{4.18}$$

such that $y(0,x') = x'$ for any $x' \in \mathbb{R}^{n-1}$, where $Y(t)$ is as in (4.13c). Let

$$\{2, \dots, n\} = \mathrm{I}'_- \cup \mathrm{I}'_+$$

be the partition such that

$$\mathrm{I}'_- = \mathrm{I}_- - \{1\} \quad \text{if } 1 \in \mathrm{I}_- \quad \text{and} \quad \mathrm{I}'_+ = \mathrm{I}_+ - \{1\} \quad \text{if } 1 \in \mathrm{I}_+.$$

It is easy to see that Eq. (4.18) also satisfies conditions (4.13a) and (4.13b) instead of I_- and I_+ by I'_- and I'_+, respectively. Thus, inductively, we assume that Eq. (4.18) possesses a unique decomposition

$$\mathbb{R}^{n-1} = \mathbb{E}'_- \oplus \mathbb{E}'_+ \tag{4.19}$$

such that there are constants $\eta_1 > 0$ and $d_1 > 0$ so that

$$\left\| y(s+T, x')\right\| \leqslant \left\| y(s, x')\right\| e^{-\eta_1 T} \quad \forall (s, x') \in \mathbb{R} \times \mathbb{E}'_- \text{ and } T \geqslant d_1 \tag{4.20a}$$

and

$$\left\| y(s+T, x')\right\| \geqslant \left\| y(s, x')\right\| e^{\eta_1 T} \quad \forall (s, x') \in \mathbb{R} \times \mathbb{E}'_+ \text{ and } T \geqslant d_1 \tag{4.20b}$$

and $\dim \mathbb{E}'_- = \operatorname{Card} \mathrm{I}'_-$ and $\dim \mathbb{E}'_+ = \operatorname{Card} \mathrm{I}'_+$. Moreover, η_1 and d_1 are both determined completely by η, d and $\tilde{\eta}$.

Next, we assert that for any $x \in \mathbb{R}^n$

$$\left\| \tilde{z}(s+t, x)\right\| \leqslant \left\| \tilde{z}(s, x)\right\| e^{\tilde{\eta} t} \quad \forall s \in \mathbb{R} \text{ and } t \geqslant 0 \tag{4.21}$$

because for $x \neq 0$ we have

$$\begin{aligned}
\log \frac{\|\tilde{z}(s+t, x)\|}{\|\tilde{z}(s, x)\|} &= \int_s^{s+t} \frac{d}{d\tau} \log\left\| \tilde{z}(\tau, x)\right\| d\tau \\
&= \int_s^{s+t} \left\langle \frac{\tilde{z}(\tau, x)}{\|\tilde{z}(\tau, x)\|}, \widetilde{B}(\tau) \frac{\tilde{z}(\tau, x)}{\|\tilde{z}(\tau, x)\|} \right\rangle d\tau.
\end{aligned}$$

To prove the claim above, we need only to consider the case $1 \in \mathrm{I}_-$; otherwise we consider the linear equation

$$\frac{d\tilde{z}}{dt} = \widetilde{B}(-t)\tilde{z}, \quad (t, \tilde{z}) \in \mathbb{R} \times \mathbb{R}^n.$$

Thus, we assume from now on that $1 \in \mathrm{I}_-$.

Let

$$\widetilde{\mathbb{E}}_- = \left\{ x = (x_1, x')^T \in \mathbb{R} \times \mathbb{R}^{n-1} \,\middle|\, x_1 \in \mathbb{R} \text{ and } x' \in \mathbb{E}'_- \right\}. \tag{4.22}$$

Clearly, $\dim \widetilde{\mathbb{E}}_- = \operatorname{Card} \mathrm{I}_-$. If $x = (x_1, x')^T \in \widetilde{\mathbb{E}}_-$ and $T \geqslant d + d_1$ then, by (4.16b), (4.17) and (4.13a), we obtain

$$\begin{aligned}
\left\| \tilde{z}(s+T, x)\right\| &\leqslant \left\| y(s+T, x')\right\| + \left| \mathrm{Pr}_1 \tilde{z}(s+T, x)\right| \\
&\leqslant \left\| y(s, x')\right\| e^{-\eta_1 T} + \left| \mathrm{Pr}_1 \tilde{z}(s+T, x)\right|
\end{aligned}$$

$$\leqslant \|\tilde{z}(s,x)\|e^{-\eta_1 T} + \|\tilde{z}(s,x)\|e^{-\eta T} + \tilde{\eta}\int_0^T f(t;s,x)\,dt, \tag{4.23}$$

where

$$f(t;s,x) = \|\mathrm{Pr}_{2\cdots n}\tilde{z}(s+t,x)\|e^{\int_t^T \tilde{b}_{11}(s+\tau)\,d\tau}$$

which is such that

$$\int_{d_1}^{T-d} f(t;s,x)\,dt \leqslant \|\tilde{z}(s,x)\| \int_{d_1}^{T-d} e^{-\eta_1-\eta(T-t)}\,dt$$
$$\leqslant \|\tilde{z}(s,x)\|(T-d-d_1)e^{-\min\{\eta_1,\eta\}T}$$

and

$$\int_0^{d_1} f(t;s,x)\,dt \leqslant \|\tilde{z}(s,x)\| \int_0^{d_1} e^{\tilde{\eta}t+\int_t^T \tilde{b}_{11}(s+\tau)\,d\tau}\,dt$$
$$\leqslant e^{\tilde{\eta}d_1}\|\tilde{z}(s,x)\|\frac{e^{-\eta T}(e^{\eta d_1}-1)}{\eta}$$

and

$$\int_{T-d}^{T} f(t;s,x)\,dt \leqslant \int_{T-d}^{T} \|y(s+t,x')\|e^{\tilde{\eta}d}\,dt$$
$$\leqslant \|y(s,x')\|e^{\tilde{\eta}d}\int_{T-d}^{T} e^{-\eta_1 t}\,dt$$
$$\leqslant e^{\tilde{\eta}d}\|\tilde{z}(s,x)\|\frac{e^{-\eta_1 T}(e^{\eta_1 d}-1)}{\eta_1}.$$

Thus, for $x \in \widetilde{\mathbb{E}}_-$ and $T \geqslant d + d_1$ we have

$$\begin{aligned}\|\tilde{z}(s+T,x)\| \leqslant{}& \|\tilde{z}(s,x)\|e^{-\eta_1 T} + \|\tilde{z}(s,x)\|e^{-\eta T}\\ &+ \|\tilde{z}(s,x)\|\biggl\{e^{-\min\{\eta_1,\eta\}T}(T-d-d_1)\tilde{\eta}\\ &+ \frac{e^{-\eta T}(e^{\eta d_1}-1)\tilde{\eta}e^{\tilde{\eta}d_1}}{\eta} + \frac{e^{-\eta_1 T}(e^{\eta_1 d}-1)\tilde{\eta}e^{\tilde{\eta}d}}{\eta_1}\biggr\}.\end{aligned} \tag{4.24}$$

On the other hand, put

$$\widetilde{\mathbb{E}}_+ = \{x = (x_1, x')^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x' \in \mathbb{E}'_+ \text{ and } x_1 = x_1(x')\},$$

where $x_1(x')$ is defined as follows: Letting

$$x_1(x')(t) = \int_{-\infty}^{t} \tilde{b}(r) y(r, x') e^{\int_r^t \tilde{b}_{11}(t')\,dt'}\,dr \quad \forall x' \in \mathbb{E}'_+, \tag{4.25}$$

$x_1(x') = x_1(x')(0)$. Note that as $\|\tilde{b}(t)\| \leqslant \tilde{\eta}$ for $t \in \mathbb{R}$ and $1 \in \mathrm{I}_-$, the integral above exists. It is easy to see that $\widetilde{\mathbb{E}}_+$ is a linear subspace of $\mathbb{R}^n$ with $\dim \mathbb{E}_+ = \operatorname{Card} \mathrm{I}_+$ and that

$$\tilde{z}(t, x) = (x_1(x')(t), y(t, x'))^T \quad \text{for } x = (x_1, x')^T \in \widetilde{\mathbb{E}}_+.$$

Now for any $x = (x_1, x')^T \in \widetilde{\mathbb{E}}_+$ and $T \geqslant d_1$ we have

$$\begin{aligned}
\|\tilde{z}(s, x)\| &\leqslant \|y(s, x')\| + |x_1(x')(s)| \\
&\leqslant \|y(s + T, x')\| e^{-\eta_1 T} + \tilde{\eta} \int_{-\infty}^{s} \|y(r, x')\| e^{\int_r^s \tilde{b}_{11}(t')\,dt'} dr \\
&\leqslant \|y(s + T, x')\| e^{-\eta_1 T} + \tilde{\eta} \int_{-\infty}^{s} \|y(T + s, x')\| e^{-\eta_1 (T + s - r) + \int_r^s \tilde{b}_{11}(t')\,dt'} dr \\
&\leqslant \|\tilde{z}(s + T, x)\| e^{-\eta_1 T} \left\{1 + \tilde{\eta} \int_{-\infty}^{s} g(r, s)\,dr\right\}
\end{aligned}$$

where

$$g(r, s) = e^{-\eta_1 (s - r) + \int_r^s \tilde{b}_{11}(t')\,dt'}$$

such that

$$\int_{-\infty}^{s-d} g(r, s)\,dr \leqslant \int_{-\infty}^{s-d} e^{-(\eta_1 + \eta)(s - r)}\,dr \leqslant \frac{1}{\eta_1 + \eta}$$

and

$$\int_{s-d}^{s} g(r, s)\,dr \leqslant d e^{\tilde{\eta} d}.$$

Therefore, for any $x = (x_1, x')^T \in \widetilde{\mathbb{E}}_+$ and $T \geqslant d_1$ we have

$$\|\tilde{z}(s,x)\| \leqslant \|\tilde{z}(s+T,x)\| e^{-\eta_1 T}\left\{1 + \tilde{\eta}\left[\frac{1}{\eta_1+\eta} + de^{\tilde{\eta} d}\right]\right\}. \tag{4.26}$$

Now, we have $\mathbb{R}^n = \widetilde{\mathbb{E}}_- \oplus \widetilde{\mathbb{E}}_+$ with $\dim \widetilde{\mathbb{E}}_- = \operatorname{Card} \mathrm{I}_-$ and $\dim \widetilde{\mathbb{E}}_+ = \operatorname{Card} \mathrm{I}_+$. From (4.24) and (4.26) it is easy to see that for

$$\eta' = \frac{1}{2}\min\{\eta, \eta_1\} \quad \text{and} \quad d' \gg d + d_1,$$

which are completely determined by η, d and $\tilde{\eta}$, the properties (4.14a) and (4.14b) both hold.

If $x \in \mathbb{R}^n - (\widetilde{\mathbb{E}}_- \cup \widetilde{\mathbb{E}}_+)$, then x can be written as $x = x_- + x_+$ where $0 \neq x_- \in \widetilde{\mathbb{E}}_-$ and $0 \neq x_+ \in \widetilde{\mathbb{E}}_+$. Then, by (4.14a) and (4.14b) we have

$$\limsup_{t\to+\infty} \|\tilde{z}(t,x)\| = +\infty \quad \text{and} \quad \limsup_{t\to-\infty} \|\tilde{z}(t,x)\| = +\infty$$

which implies that the direct decomposition $\mathbb{R}^n = \widetilde{\mathbb{E}}_- \oplus \widetilde{\mathbb{E}}_+$ satisfying (4.14a) and (4.14b) is unique.

Letting $\mathbb{E}_-(\lambda) = \widetilde{\mathbb{E}}_-$ and $\mathbb{E}_+(\lambda) = \widetilde{\mathbb{E}}_+$, as $z(s+t,x) = e^{\lambda(s+t)}\tilde{z}(s+t,x)$ for any $x \in \mathbb{R}^n$ it follows from (4.14a) and (4.14b) that (4.10a) and (4.10b) hold.

By induction on n and (4.22) it is easy to see that if $\mathrm{I}_- = \{1, \ldots, \operatorname{Card} \mathrm{I}_-\}$ then $\mathbb{E}_-(\lambda) = \mathbb{R}^{\operatorname{Card} \mathrm{I}_-} \times \{0_{n-\operatorname{Card} \mathrm{I}_-}\}$.

Thus, the proof of Lemma 4.2 is complete. □

Let $\mathrm{gl}_\triangle(n,\mathbb{R})$ be the space of $n \times n$, real, upper-triangular matrices with the norm $\|\cdot\|$ to be defined by the manner as in (4.8). To any continuous

$$B : W \to \mathrm{gl}_\triangle(n,\mathbb{R}),$$

there is a corresponding random linear system

$$\frac{dz}{dt} = B(t.w)z, \quad (t,z) \in \mathbb{R} \times \mathbb{R}^n;\ w \in W,$$

based on (W,θ). We also call such B a linear cocycle over (W,θ).

We conclude easily from Lemmas 3.1 and 4.2 the following result.

Corollary 4.3. *Let (W,θ) be uniquely ergodic with ν being the unique θ-invariant probability measure on W. Then, the set, consisting of all linear cocycles B which is uniformly hyperbolic with simple Lyapunov spectrum over (W,θ), is open and dense in $C^0(W, \mathrm{gl}_\triangle(n,\mathbb{R}^n))$ under the topology induced by the uniform $\|\cdot\|$-norm.*

Proof. From [17, Theorem 4.6] it follows that

$$\mathfrak{H} = \{B \in C^0(W, \mathrm{gl}_\triangle(n,\mathbb{R}^n)) \mid B \text{ is nonuniformly hyperbolic and simple spectral}\}$$

is open and dense. As (W,θ) is uniquely ergodic, it follows from Oxtoby's uniform ergodic theorem [38] that for any B

$$\lim_{T\to\pm\infty}\frac{1}{T}\int_0^T B_{kk}(t.w)\,dt=\int_W B_{kk}\,d\nu,\quad k=1,\dots,n,$$

for any $w\in W$ and uniformly on W. Then, we obtain from Lemma 4.2 that $B\in\mathfrak{H}$ is uniformly hyperbolic.

Thus, the proof is completed. □

5. Proofs of Theorems 1 and 2

This section is devoted to the proofs of Theorems 1 and 2.

5.1. Proof of Theorem 1

We now can finish the proof of our main result Theorem 1 by using Lemmas 3.2, 4.1 and 4.2.

Let (θ,Θ) and $\hat{\lambda}$ be as in Theorem 1. Without loss of generality, we assume as in Section 2.1 that there are constants $\varrho'>0$ and $C'>0$ such that

$$\left\|\Theta(t+\bar{t},w)\cdot x\right\|\leqslant C'e^{t(\hat{\lambda}-\varrho')}\left\|\Theta(\bar{t},w)\cdot x\right\|\quad\forall x\in\mathbb{E}^s(w;\hat{\lambda})$$

for all $\bar{t}\in\mathbb{R}$, $t\geqslant 0$ and for a.e. $w\in W$. Let Γ and $\Bbbk$ be defined as in Section 2.1.

First, we obtain the following result.

Lemma 5.1. *There exists a unique direct decomposition of $\mathbb{R}^n$ into subspaces*

$$\mathcal{D}_{\hat{\lambda}}:\Gamma\ni w\mapsto\mathbb{E}_-(w;\hat{\lambda})\oplus\mathbb{E}_+(w;\hat{\lambda})=\mathbb{R}^n(w)$$

such that there are constants $\varrho>0$ and $T'>0$ so that

$$\left\|\Theta(T+t,w)\cdot x\right\|\leqslant e^{T(\hat{\lambda}-\varrho)}\left\|\Theta(t,w)\cdot x\right\|\quad\forall x\in\mathbb{E}_-(w;\hat{\lambda})$$

and

$$\left\|\Theta(T+t,w)\cdot y\right\|\geqslant e^{T(\hat{\lambda}+\varrho)}\left\|\Theta(t,w)\cdot y\right\|\quad\forall y\in\mathbb{E}_+(w;\hat{\lambda})$$

for all $t\in\mathbb{R}$ and for any $T\geqslant T'$.

Proof. It follows from Lemma 3.2 that there exist two constants $\eta>0$ and $d>0$ such that

$$\frac{1}{T}\int_0^T\omega_i\big(\big(\theta,\widehat{\Theta}_n^\sharp\big)(t+\bar{t},w,\vec{\gamma})\big)\,dt\leqslant\hat{\lambda}-\eta\quad(1\leqslant i\leqslant\Bbbk)$$

and

$$\frac{1}{T}\int_0^T \omega_i\big(\big(\theta, \widehat{\Theta}_n^\sharp\big)(t+\bar{t}, w, \vec{\gamma})\big)\,dt \geqslant \hat{\lambda}+\eta \quad (\Bbbk < i \leqslant n \text{ if } \Bbbk < n)$$

for all $(w, \vec{\gamma}) \in \mathcal{F}^\sharp(\Gamma)$ and for any $\bar{t} \in \mathbb{R}$, $T \geqslant d$. Put

$$\mathbb{b} = \sup_{(w,\vec{\gamma})\in W\times\mathscr{F}_n^\sharp} \big\{\big\|R(\vec{\gamma}_w)\big\|\big\}. \tag{5.1}$$

Then $\mathbb{b} < \infty$ from Lemma 4.1.

Now, given any $w \in \Gamma$ and any $(w, \vec{\gamma}) \in \mathcal{F}^\sharp(\Gamma)$. Let $z = z_{\vec{\gamma}_w}(t, x)$ be the solution of Eq. $(R_{\vec{\gamma}_w})$

$$\frac{dz}{dt} = R\big(t.(w, \vec{\gamma})\big)z \tag{5.2}$$

satisfying $z_{\vec{\gamma}_w}(0, x) = x$ for any $x \in \mathbb{R}^n$.

Thus, from Lemmas 4.1 and 4.2 there exists a unique direct decomposition of $\mathbb{R}^n$ into subspaces

$$\mathbb{R}^n = \mathbb{E}_-(\hat{\lambda}) \oplus \mathbb{E}_+(\hat{\lambda})$$

such that there are two constants $\varrho > 0$ and $T' > 0$ which both are completely determined by $\hat{\lambda}, \eta, d$ and $\mathbb{b}$, so that

$$\big\|z_{\vec{\gamma}_w}(t+T, x)\big\| \leqslant \big\|z_{\vec{\gamma}_w}(t, x)\big\|e^{(\hat{\lambda}-\varrho)T} \quad \forall (t, x) \in \mathbb{R}\times\mathbb{E}_-(\hat{\lambda}) \text{ and } T \geqslant T' \tag{5.3}$$

and

$$\big\|z_{\vec{\gamma}_w}(t+T, x)\big\| \geqslant \big\|z_{\vec{\gamma}_w}(t, x)\big\|e^{(\hat{\lambda}+\varrho)T} \quad \forall (t, x) \in \mathbb{R}\times\mathbb{E}_+(\hat{\lambda}) \text{ and } T \geqslant T' \tag{5.4}$$

and $\dim\mathbb{E}_-(\hat{\lambda}) = \Bbbk$ and $\dim\mathbb{E}_+(\hat{\lambda}) = n - \Bbbk$.

Next, we put

$$\mathbb{E}_-(w; \hat{\lambda}) = \mathcal{T}_{\vec{\gamma}}\big(\mathbb{E}_-(\hat{\lambda})\big) \quad \text{and} \quad \mathbb{E}_+(w; \hat{\lambda}) = \mathcal{T}_{\vec{\gamma}}\big(\mathbb{E}_+(\hat{\lambda})\big), \tag{5.5}$$

where $\mathcal{T}_{\vec{\gamma}}$ is defined as in (4.1). Let

$$\mathcal{D}_{\hat{\lambda}} : \Gamma \ni w \mapsto \mathbb{E}_-(w; \hat{\lambda}) \oplus \mathbb{E}_+(w; \hat{\lambda}) = \mathbb{R}^n(w).$$

Then we have $\mathbb{R}^n(w) = \mathbb{E}_-(w; \hat{\lambda}) \oplus \mathbb{E}_+(w; \hat{\lambda})$, and by Lemma 4.1 again we obtain

$$\big\|\Theta(T+t, w)\cdot x\big\| \leqslant e^{T(\hat{\lambda}-\varrho)}\big\|\Theta(t, w)\cdot x\big\| \quad \forall x \in \mathbb{E}_-(w; \hat{\lambda})$$

and

$$\big\|\Theta(T+t, w)\cdot y\big\| \geqslant e^{T(\hat{\lambda}+\varrho)}\big\|\Theta(t, w)\cdot y\big\| \quad \forall y \in \mathbb{E}_+(w; \hat{\lambda})$$

for all $t \in \mathbb{R}$ and for any $T \geqslant T'$.

The uniqueness of the splitting $\mathbb{R}^n(w) = \mathbb{E}_-(w; \hat{\lambda}) \oplus \mathbb{E}_+(w; \hat{\lambda})$ is evident.

The proof of Lemma 5.1 is thus completed. □

Proof of Theorem 1. Let ϱ and T' be defined as in Lemma 5.1. We easily see from the Oseledets multiplicative ergodic theorem and Lemma 5.1 that

$$\mathbb{E}^s(w; \hat{\lambda}) = \mathbb{E}_-(w; \hat{\lambda}) \quad \text{and} \quad \mathbb{E}^u(w; \hat{\lambda}) = \mathbb{E}_+(w; \hat{\lambda}) \quad \forall w \in \Gamma.$$

Since $(t, w) \mapsto \Theta(t, w)$ is continuous on $[-T', T'] \times W$, there exists some constant $C \geqslant 1$ such that

$$\left\|\Theta(t, w) \cdot x\right\| \leqslant Ce^{(\hat{\lambda}-\varrho)t}\|x\| \quad \forall (t, w, x) \in [0, T'] \times W \times \mathbb{R}^n$$

and

$$\left\|\Theta(t, w) \cdot y\right\| \leqslant Ce^{(\hat{\lambda}+\varrho)t}\|y\| \quad \forall (t, w, y) \in [-T', 0] \times W \times \mathbb{R}^n.$$

Thus, from the co-cycle property of $\Theta(t, w)$ we obtain

$$\left\|\Theta(t + t', w) \cdot x\right\| \leqslant Ce^{(\hat{\lambda}-\varrho)t}\left\|\Theta(t', w) \cdot x\right\| \quad \forall x \in \mathbb{E}^s(w; \hat{\lambda})$$

for all $t' \in \mathbb{R}$, $t \geqslant 0$ and for $w \in \Gamma$, and

$$\left\|\Theta(t + t', w) \cdot y\right\| \leqslant Ce^{(\hat{\lambda}+\varrho)t}\left\|\Theta(t', w) \cdot y\right\| \quad \forall y \in \mathbb{E}^u(w; \hat{\lambda})$$

for all $t' \in \mathbb{R}$, $t \leqslant 0$ and for $w \in \Gamma$.

Therefore, the proof of Theorem 1 is complete. □

5.2. Proof of Theorem 2

We now prove Theorem 2 stated in Section 1.1.

Proof. The statement (1) is evident from Theorem 1 and the multiplicative ergodic theorem. Indeed, based on the fact that $\dim E(w)$ is constant for a.e. $w \in M$ we let $\Bbbk = \dim E(w)$ for a.e. w. Let $\Gamma(f)$ be the Oseledets regular point set of f. Note here that $\overline{\Gamma(f)} = M$. For any $w \in \Gamma(f)$, let

$$\chi_1(w) \leqslant \cdots \leqslant \chi_{\Bbbk}(w) \leqslant \chi_{\Bbbk+1}(w) \leqslant \cdots \leqslant \chi_n(w)$$

be the Lyapunov exponents of f at w counting with multiplicity. Then, by the assumption of Theorem 2 we get

$$\chi_1(w) \leqslant \cdots \leqslant \chi_{\Bbbk}(w) \leqslant \ln\lambda < \chi_{\Bbbk+1}(w) \leqslant \cdots \leqslant \chi_n(w) \quad \forall w \in \Gamma(f) \tag{5.6}$$

and further by the Oseledets ergodic theorem

$$E(w) = \bigoplus_{\lambda_i(w) \leqslant \ln\lambda} \mathbb{E}_i(w) \quad \text{and} \quad F(w) = \bigoplus_{\lambda_i(w) > \ln\lambda} \mathbb{E}_i(w) \quad \forall w \in \Gamma(f) \tag{5.7a}$$

$$\Gamma(f) \ni w \mapsto E(w) \text{ is continuous.} \tag{5.7b}$$

Now by Lemmas 2.3 and 3.1 and (2.11) we can find some $\eta > 0$, which satisfies $\hat{\lambda} = \eta/2 + \ln\lambda$, such that

$$\lim_{\ell\to\infty} \frac{1}{\ell} \ln\left\| D_w f^\ell \cdot \vec{v} \right\| \geqslant \hat{\lambda} + \eta/2, \quad \vec{v} \in T_w M \setminus E(w),$$

and

$$\lim_{\ell\to\infty} \frac{1}{\ell} \ln\left\| D_w f^\ell \cdot \vec{v} \right\| \leqslant \hat{\lambda} - \eta/2, \quad \vec{v} \in E(w),$$

for a.e. $w \in M$.

The statement (2) is evident too. In fact, put

$$W_k = \left\{ w \in \Gamma \mid \dim E(w) = k \right\}, \quad k = 0, 1, \ldots, \dim M,$$

where Γ is the Oseledets regular point set of f on M. It follows from the continuity of the distribution $\mathbb{D}(w)$ that W_k are f-invariant closed subsets of Γ, where we think of Γ as a subspace of M. Then

$$\nu(\overline{W}_k - W_k) = 0 \quad \forall \nu \in \mathcal{M}_{\mathrm{erg}}(M, f),$$

where $\overline{W}_k$ denotes the closure in M. We now apply Theorem 1 to the linear skew-product system

$$Df : T_{\overline{W}_k} M \to T_{\overline{W}_k} M; \quad (w, \vec{v}) \mapsto \left(f(w), D_w f \cdot \vec{v} \right) \quad \forall \vec{v} \in T_w M, w \in \overline{W}_k$$

and then obtain the required result from the statement (1) proved.

Thus, the proof of Theorem 2 is completed. □

6. Hyperbolicity of $GL(2, \mathbb{R})$-valued cocycles

In this section, we shall prove Theorem 3 and the open-and-dense hyperbolicity of $GL(2, \mathbb{R})$-valued cocycles *based on an endomorphism* (continuous and surjective, but not necessarily injective)

$$\theta : W \to W$$

which generates a semi-dynamical system

$$\theta : \mathbb{Z}_+ \times W \to W; \quad (t, w) \mapsto t.w$$

where $t.w = \theta^t w = (\overbrace{\theta \circ \cdots \circ \theta}^{t\text{-times}})w$. In the proof of these results, we need to use the natural extension of a cocycle and existence of 'positive core' (Lemma 6.2 below). Here the notation 'positive core' is borrowed from Furman [23].

6.1. Positive core

Given a positive integer $n \geqslant 2$ and let $\mathrm{gl}(n,\mathbb{R})$ be the space of all real n-by-n matrices with the standard topology induced by the $\|\cdot\|$-norm as in (4.8). For any continuous random matrix $L: W \to \mathrm{gl}(n,\mathbb{R})$ and for any homeomorphism $T: W \to W$, we consider existence of invariant nontrivial section $\xi_L: W \to \mathbb{R}^n$ by the action of the skew-product discrete flow:

$$(T,L): W \times \mathbb{R}^n \to W \times \mathbb{R}^n; \quad (w,x) \mapsto \big(Tw, L(w)\cdot x\big).$$

We denote by Δ the standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$ defined by

$$\Delta = \left\{ u \in \mathbb{R}^n \,\Big|\, \sum_1^n u_i = 1,\ u_i \geqslant 0 \right\}.$$

For convenience, we choose for Δ the angle metric given by

$$\mathrm{dist}(u,w) = \cos^{-1}\left|\big\langle u/\|u\|, w/\|w\|\big\rangle\right| \quad u,w \in \Delta.$$

For any positive random matrix L (i.e. $L_{ij}(w) > 0\ \forall w \in W,\ \forall 1 \leqslant i,j \leqslant n$) we have

$$\hat{L}(w): \Delta \to \Delta; \quad u \mapsto r(w,u)L(w)\cdot u$$

where $r(w,u)$ is the unique positive real number such that $r(w,u)L(w)\cdot u \in \Delta$. It is easy to see that $r: W \times \Delta \to (0,\infty)$ is continuous with respect to (w,u) in $W \times \Delta$.

Then, for any positive continuous random matrix L we obtain over T a naturally induced skew-product flow:

$$(T,\hat{L}): W \times \Delta \to W \times \Delta; \quad (w,u) \mapsto \big(Tw, \hat{L}(w)\cdot u\big).$$

In order to prove the existence of positive core, we need a lemma which is a generalization of [23, Lemma 5].

Lemma 6.1. *Let $L(t) \in \mathrm{gl}(n,\mathbb{R})$, $t = 1,2,\dots,$ be a sequence of positive matrices, bounded in the sense that there exists some $\mathfrak{b} > 0$ such that $\mathfrak{b} < L_{ij}(t) < \mathfrak{b}^{-1}$ for all $t \geqslant 1$ and $1 \leqslant i,j \leqslant n$. Then there exists a unique point $\bar{u} \in \Delta$ such that*

$$\{\bar{u}\} = \bigcap_{k=1}^{\infty} \hat{L}(1) \circ \cdots \circ \hat{L}(k)(\Delta)$$

and $\bar{u}$ is an interior point of Δ; i.e. $\bar{u}_i > 0$ for $i = 1,\dots,n$.

Proof. The argument is almost the same as the proof of [23, Lemma 5], but for the completeness, we give the details.

Since $L(t)$ is linear and positive on Δ, $\hat{L}(1) \circ \cdots \circ \hat{L}(t)$ preserves the four points cross ratios

$$[u;v;w;z] = \frac{\|u-w\| \cdot \|v-z\|}{\|u-z\| \cdot \|v-w\|}$$

provided that u, v, w, z lie on the same line in Δ. Now let

$$K = \bigcap_{1}^{\infty} K_t,$$

where $K_t = \hat{L}(1) \circ \cdots \circ \hat{L}(t)(\Delta)$ form a descending sequence of convex compacts. Assume that K is not a singleton, and let $u \neq v$ be two extremal points of K. Let w_t, z_t be the intersection of the line (u, v) with the boundary ∂K_t. Notice here that $\hat{L}(1) \circ \cdots \circ \hat{L}(t) : \Delta \to \Delta$ is injective, since $L(t)|\mathbb{R}^n_+$ is injective by $L_{ij}(t) > 0$ for each $t = 1, 2, \ldots$. Thus, we let $w'_t, z'_t, u'_t, v'_t \in \Delta$ be the preimages of w_t, z_t, u, v. Then, w'_t, z'_t lie in $\partial\Delta$, but $u'_t, v'_t \in \hat{L}(t+1)(\Delta)$ since $u, v \in K_{t+1} = \hat{L}(1) \circ \cdots \circ \hat{L}(t)(\hat{L}(t+1)(\Delta))$. The $\mathfrak{S}$-boundedness of $L(t+1)$ implies that $\hat{L}(t+1)(\Delta)$ is uniformly separated from $\partial\Delta$. Thus, the cross ratio $[u'_t, v'_t; w'_t, z'_t]$ is bounded from 0 and ∞. On the other hand, $w_t \to u$ and $z_t \to v$ imply $[u, v; w_t; z_t] \to 0$ as $t \to \infty$, causing the contradiction. □

We now prove the existence of positive core for positive linear cocycles, which is an important technical lemma for the proof of Theorem 4.

Lemma 6.2. *Let $L : W \to \mathrm{gl}(n, \mathbb{R})$ be continuous and positive. Then, there exists a unique continuous random vector, called the "positive core" of (T, L),*

$$\xi_L : W \to \Delta$$

such that

$$\xi_L(w) > 0, \quad L(w) \cdot \xi_L(w) = r\big(w, \xi_L(w)\big)\xi_L(Tw) \quad \forall w \in W \tag{6.1a}$$

and

$$\frac{1}{t} \log\big\| L\big(T^{t-1}w\big) \circ \cdots \circ L(w) \cdot \xi_L(w)\big\| = \frac{1}{t} \sum_{i=0}^{t-1} \log r\big(T^i w, \xi_L\big(T^i w\big)\big) \quad \forall t \in \mathbb{N}. \tag{6.1b}$$

Proof. Since W is compact and L is continuous and positive, there is some $\mathfrak{S} > 0$ such that $\mathfrak{S} < L_{ij}(w) < \mathfrak{S}^{-1}$ for all $w \in W$ and $1 \leqslant i, j \leqslant n$.

Now we consider $(T, \hat{L}) : W \times \Delta \to W \times \Delta$, we easily get that the compact subset

$$Q = \bigcap_{t=1}^{\infty} (T, \hat{L})^t (W \times \Delta) \subset W \times \Delta$$

is nonempty. We next claim that Q is a graph of some continuous function

$$\bar{u} : W \to \Delta.$$

Indeed, for any fixed $w \in W$ the fiber Q_w of Q is given by

$$Q_w = \bigcap_{t=1}^{\infty} (T, \hat{L})^t \left(\left\{ T^{-t} w \right\} \times \Delta \right) \\ = \{w\} \times \bigcap_{t=1}^{\infty} \hat{L}\left(T^{-1} w\right) \circ \cdots \circ \hat{L}\left(T^{-t} w\right)(\Delta)$$

and, by Lemma 6.1 for the case $L(t) = L(T^{-t} w)$, Q_w consists of a single point $(w, \bar{u}(w))$. Since $Q = \{(w, \bar{u}(w)) \mid w \in W\}$ is closed, the function $\bar{u}(w)$ is continuous, and $(T, \hat{L})$-invariance of Q implies that

$$\bar{u}(Tw) = \hat{L}(w) \cdot \bar{u}(w) \quad \forall w \in W.$$

We thus proved the claim above.

Next, put

$$\xi_L(w) = \bar{u}(w) \quad \forall w \in W.$$

It is easy to see that $\xi_L(w)$ satisfies the requirement of Lemma 6.2 and such ξ_L is uniquely defined by Q.

Thus, the proof is completed. □

We may think of $\xi_L(w)$ as a random Perron eigenvector associated to the random eigenvalue $\rho_L(w) = r(w, \xi_L(w))$ based on (T, L).

6.2. Natural extensions

Since θ need not to be invertible, for a random positive matrix it is possible that the dimension of positive random eigenvector space is greater than 1, and hence we cannot take a maximal random eigenvalue. This causes that we cannot apply the Birkhoff ergodic theorem to obtain the desired uniformity. To overcome this point, we consider the natural extension of (W, θ).

Let $W_\theta = \{\underline{w} = (w_i)_{-\infty}^{+\infty} \in \prod_{-\infty}^{+\infty} W \mid \theta(w_i) = w_{i+1} \ \forall i \in \mathbb{Z}\}$ and define the shift

$$\sigma_\theta : W_\theta \to W_\theta$$

by

$$\sigma_\theta(\underline{w})_i = w_{i+1} \quad \forall \underline{w} = (w_i).$$

Under the usual metric

$$\mathrm{dist}(\underline{w}, \underline{w}') = \sum_{-\infty}^{+\infty} 2^{-|i|} \mathrm{dist}(w_i, w_i') \quad \forall \underline{w} = (w_i), \underline{w}' = (w_i') \in W_\theta,$$

σ_θ is a homeomorphism of the compact metric space W_θ. Put

$$\pi_0 : W_\theta \to W; \quad \underline{w} = (w_i) \mapsto w_0.$$

It is easy to see that π_0 is a semi-conjugacy from $(W_\theta, \sigma_\theta)$ onto (W, θ). Since for any $\underline{w}, \underline{w}' \in \pi_0^{-1}(w)$ for any given $w \in W$

$$\operatorname{dist}\bigl(\sigma_\theta^k \underline{w}, \sigma_\theta^k \underline{w}'\bigr) \to 0 \quad \text{as } k \to +\infty,$$

we easily get

Lemma 6.3. *If (W, θ) is uniquely ergodic, then the natural extension $(W_\theta, \sigma_\theta)$ is uniquely ergodic as well.*

We next naturally extend a linear cocycle over (W, θ). For any $A : W \to \mathrm{gl}(n, \mathbb{R})$, define the natural extension

$$A_\theta : W_\theta \to \mathrm{gl}(n, \mathbb{R})$$

in the way

$$A_\theta(\underline{w}) = A(w_0), \quad \text{where } w_0 = \pi_0(\underline{w}) \text{ for any } \underline{w} \in W_\theta.$$

Clearly, if A is continuous then A_θ is continuous too and we have for any $t \geqslant 1$ and for any $\underline{w} = (w_i) \in W_\theta$

$$\begin{aligned} \text{over } \theta \qquad A(t, w_0) &= A\bigl(\theta^{t-1} w_0\bigr) \circ \cdots \circ A(w_0) \\ &= A_\theta\bigl(\sigma_\theta^{t-1} \underline{w}\bigr) \circ \cdots \circ A_\theta(\underline{w}) \\ &= A_\theta(t, \underline{w}) \qquad\qquad \text{over } \sigma_\theta. \end{aligned}$$

This shows together with Lemma 6.3 the following

Lemma 6.4. *The linear cocycle $A \in C^0(W, \mathrm{gl}(n, \mathbb{R}))$ over (W, θ) has the same Lyapunov exponents as A_θ over $(W_\theta, \sigma_\theta)$.*

Moreover, the following two lemmas are also evident.

Lemma 6.5. *Given any $A \in C^0(W, \mathrm{gl}(n, \mathbb{R}))$. If*

$$\frac{1}{t} \log\bigl\|A_\theta(t, \underline{w})\bigr\| \to \lambda(A_\theta) \quad \text{as } t \to +\infty$$

uniformly for all $\underline{w} \in W_\theta$ over $(W_\theta, \sigma_\theta)$, then

$$\frac{1}{t} \log\bigl\|A(t, w)\bigr\| \to \lambda(A)\bigl(= \lambda(A_\theta)\bigr) \quad \text{as } t \to +\infty$$

uniformly for all $w \in W$ over (W, θ).

Proof. The statement comes easily from definitions. $\square$

Lemma 6.6. *If $\theta : W \to W$ is equicontinuous, then $\sigma_\theta : W_\theta \to W_\theta$ is also equicontinuous.*

Proof. The result follows easily from the fact $\sigma_\theta^k = \sigma_{\theta^k}$ for any $k \in \mathbb{N}$. □

6.3. Proof of Theorem 3

In this subsection, we will prove Theorem 3 by verifying a more general result Theorem 6.7 below.

Theorem 6.7. *Let $\theta : W \to W$ be an endomorphism and assume $A : W \to GL(n, \mathbb{R})$ is a positive continuous random matrix over (W, θ). If (θ, A) is almost nonuniformly hyperbolic with* $\operatorname{Codim} \mathbb{E}^s(w) = 1$ *a.e., then (θ, A) is almost uniformly hyperbolic on W.*

Proof. Let $(W_\theta, \sigma_\theta)$ and A_θ be respectively the natural extensions of (W, θ) and A as in Section 6.2. For any $\underline{w} = (w_i) \in W_\theta$, put $\mathbb{E}^s(\underline{w}) = \mathbb{E}^s(w_0)$ and $\mathbb{E}^u(\underline{w}) = \mathbb{E}^u(w_0)$. Then, it is easy to check that $(\sigma_\theta, A_\theta)$ is also almost nonuniformly hyperbolic associated with

$$\underline{w} \mapsto \mathbb{E}^s(\underline{w}) \oplus \mathbb{E}^u(\underline{w}) \quad \text{a.e. on } W_\theta.$$

Using Lemma 6.2, it follows that there exists a random positive eigenvector

$$\xi_{A_\theta} : W_\theta \to \operatorname{Int}(\Delta) = \left\{ u \in \mathbb{R}^n \;\middle|\; \sum u_i = 1,\ u_i > 0 \right\}$$

and a corresponding random eigenvalue

$$\rho_{A_\theta} : \Omega_\theta \mapsto r\big(\underline{w}, \xi_{A_\theta}(\underline{w})\big)$$

such that

$$A_\theta(\underline{w}) \cdot \xi_{A_\theta}(\underline{w}) = \rho_{A_\theta}(\underline{w}) \xi_{A_\theta}(\sigma_\theta \underline{w}) \quad \forall \underline{w} \in W_\theta.$$

Since the uniform positivity of $A_\theta(\underline{w})$ and $\xi_{A_\theta}(\underline{w})$, there is some constant $c > 0$ such that

$$\begin{aligned} \big\| A_\theta\big(\sigma_\theta^{k-1}\underline{w}\big) \circ \cdots \circ A_\theta(\underline{w}) \big\| &\leqslant c \big\| A_\theta\big(\sigma_\theta^{k-1}\underline{w}\big) \circ \cdots \circ A_\theta(\underline{w}) \cdot \xi_{A_\theta}(\underline{w}) \big\| \\ &\leqslant c \big\| A_\theta\big(\sigma_\theta^{k-1}\underline{w}\big) \circ \cdots \circ A_\theta(\underline{w}) \big\| \end{aligned}$$

for any $\underline{w} \in W_\theta$ and $k \in \mathbb{N}$. Thus, for any $\mu_\theta \in \mathcal{M}_{\mathrm{erg}}(W_\theta, \sigma_\theta)$

$$\int_{W_\theta} \log \rho_{A_\theta} \, d\mu_\theta = \lambda_{\max}(A_\theta, \mu_\theta)$$

is just the maximal Lyapunov exponent of $(\sigma_\theta, \mu_\theta; A_\theta)$. As from the nonuniform hyperbolicity we have $\lambda_{\max}(A_\theta, \mu_\theta) > 0$ for any $\mu_\theta \in \mathcal{M}_{\mathrm{erg}}(W_\theta, \sigma_\theta)$, it follows from Lemmas 3.1 and 6.2 that there are some $\eta > 0$ and $T_0 > 0$ such that

$$\big\| A_\theta(t + t', \underline{w}) \cdot \xi_{A_\theta}(\underline{w}) \big\| \geqslant \big\| A_\theta(t', \underline{w}) \cdot \xi_{A_\theta}(\underline{w}) \big\| \exp(t\eta)$$

for all $t' \in \mathbb{R}, t \geqslant T_0$ and for any $\underline{w} \in W_\theta$. Thus, $(\sigma_\theta, A_\theta)$ is semi-hyperbolic with uniformly expanding directions $\mathbb{E}^u(\underline{w}) = \mathbb{E}(\xi_{A_\theta}(\underline{w}))$.

Thus, Theorem 6.7 follows from Theorem 1. □

6.4. Proof of Theorem 4

In what follows, we assume that $\theta : W \to W$ is a homeomorphism of W. We now can conclude the statement (1) of Theorem 4 from Lemma 6.2 and Theorem 3 very easily.

Theorem 6.8. *If (W, θ) is uniquely ergodic with ν being the unique θ-invariant probability measure on W, the set of $GL_+(2, \mathbb{R})$-valued cocycles, which are either uniformly hyperbolic or uniformly expanding or uniformly contracting on the support of ν, is open and dense in $C^0(W, GL_+(2, \mathbb{R}))$.*

Proof. Let ν be the unique θ-invariant probability measure on W. By a simple perturbation such as $A \circ e^{\epsilon I_2}, \epsilon > 0$, we obtain that linear positive cocycles having no zero exponent are dense in $C^0(W, GL_+(2, \mathbb{R}))$ based on $(W, \theta; \nu)$. □

To prove the theorem, it is sufficient to show the following Lemma 6.9.

Lemma 6.9. *If $A : W \to GL(2, \mathbb{R})$ is positive and has no zero exponent based on the unique ergodic compact system $(W, \theta; \nu)$, then A is either uniformly hyperbolic or uniformly expanding or uniformly contracting on* $\operatorname{supp}(\nu)$.

Proof. We assume that $A \in C^0(W, GL(2, \mathbb{R}))$ is positive and nonuniformly hyperbolic. Then, from Theorem 3 it follows easily that (θ, A) is uniformly hyperbolic on $\operatorname{supp}(\nu)$. Write

$$\mathsf{Sp}(A; \nu) = \{\lambda_1, \lambda_2\} \quad (\lambda_1 \leqslant \lambda_2)$$

the set of all Lyapunov exponents counting with multiplicity. If $\lambda_1 > 0$ or $\lambda_2 < 0$, then from [19, Corollary 1.3] we see that A is uniformly expanding or uniformly contracting on W based on (W, θ).

Thus, the proof of Lemma 6.9 is completed. □

We next prove the second part of Theorem 4. First, we need two lemmas.

Lemma 6.10. *Let $(W, \theta; \nu)$ be ergodic. Define the functional $\lambda : C^0(W, GL(2, \mathbb{R})) \to \mathbb{R}$ by*

$$\lambda(A) = \lim_{t \to \infty} \frac{1}{t} \int_W \log \| A(t, w) \| \, d\nu(w).$$

Then the set of all continuity points A of $\lambda(\cdot)$ is a dense G_δ set in $C^0(W, GL(2, \mathbb{R}))$.

Proof. The result comes from the upper semi-continuity of $\lambda(A)$ and [8, Theorem 5] and the completeness of the metric space $(C^0(W, GL(2, \mathbb{R})), \operatorname{dist}(\cdot, \cdot))$. □

The following is a generalization of [46, Proposition 2].

Lemma 6.11. *Let (W, θ) be uniquely ergodic with ν being the unique θ-invariant probability measure on W. If $A \in C^0(W, GL(2, \mathbb{R}))$ is uniform and nonuniformly hyperbolic, then A is uniformly hyperbolic on the support of ν based on (W, θ).*

Proof. The statement comes immediately from [23, Theorems 3 and 4], Lemmas 3.1 and 6.9. □

Now, the following theorem is equivalent to the second part of Theorem 4.

Theorem 6.12. *Let (W, θ) be equicontinuous and uniquely ergodic with ν being the unique θ-invariant probability measure on W. Then the set of $GL(2, \mathbb{R})$-valued cocycles based on (W, θ), which are either uniformly hyperbolic or uniformly expanding or uniformly contracting on the support of ν, is open and dense in the space $C^0(W, GL(2, \mathbb{R}))$.*

Proof. By virtue of [23, Theorem 5] and Lemma 6.10, there is a dense G_δ-set of uniform $GL(2, \mathbb{R})$-cocycles in $C^0(W, GL(2, \mathbb{R}))$.

Given any $A : W \to GL(2, \mathbb{R})$. If A is uniform based on (W, θ), then for any $\delta \in \mathbb{R}$, $e^\delta A$ is also uniform. Thus, we can assert from Lemma 6.10 and [23, Theorem 5] that there is a dense subset of $C^0(W, GL(2, \mathbb{R}))$ consisting of uniform cocycle A having no any zero Lyapunov exponents and furthermore we obtain from Lemma 6.11 that there is a dense set consisting of uniformly hyperbolic or uniformly expanding or uniformly contracting $GL(2, \mathbb{R})$-cocycles based on (W, θ). □

We thus proved Theorem 4.

Remark 6.13. If θ is only an endomorphism with a uniquely ergodic probability measure ν, then from Theorem 6.7 and Lemma 6.6 it easily follows that the results of Theorem 4 still hold.

6.5. A generalization of a theorem of P. Walters

Let $T : W \to W$ be a uniquely ergodic homeomorphism of a compact metrizable space W with T-invariant probability measure m. If $B : W \to GL(n, \mathbb{R})$ is continuous and positive, then P. Walters [44] proved that

$$\frac{1}{k} \log \left\| B\left(T^{k-1} w\right) \circ \cdots \circ B(w) \right\| \to \lambda(B) \quad \text{as } k \to +\infty$$

uniformly for $w \in W$. From Lemmas 6.2 and 6.5, we easily obtain the following slight generalization by different approaches.

Theorem 6.14. *Let $\theta : W \to W$ be an endomorphism of W with ν being the unique θ-invariant probability measure on W. If $B : W \to \mathrm{gl}(n, \mathbb{R})$ is continuous and positive, then*

$$\frac{1}{k} \log \left\| B\left(\theta^{k-1} w\right) \circ \cdots \circ B(w) \right\| \to \lambda_{\max}(B) \quad \textit{as } k \to +\infty$$

uniformly for all $w \in W$.

Proof. By virtue of Lemmas 6.5 and 6.3, without loss of generality, we assume that θ is a homeomorphism of W which is uniquely ergodic. Let $\xi_B(w)$ and $r(w, \xi_B(w))$ are the random vector and function defined as in Lemma 6.2.

Because of the uniform positivity of $B(w)$ and $\xi_B(w)$, there is some constant $c > 0$ such that

$$\begin{aligned}\|B(\theta^{k-1}w) \circ \cdots \circ B(w)\| &\leqslant c\|B(\theta^{k-1}w) \circ \cdots \circ B(w) \cdot \xi_B(w)\| \\ &\leqslant c\|B(\theta^{k-1}w) \circ \cdots \circ B(w)\|\end{aligned}$$

for any $w \in W$ and $k \in \mathbb{N}$. Thus, (6.1b) follows the required uniformity. □

7. $SL(2,\mathbb{R})$-valued cocycles

This section is devoted to prove the C^0-dense hyperbolicity theorem of $SL(2,\mathbb{R})$-valued cocycles based on a minimal subshift satisfying the Boshernitzan condition.

7.1. Boshernitzan condition

We recall some further notions. From now on, let

$$\sigma : \Sigma_k \to \Sigma_k; \quad \sigma(w)_t = w_{t+1}$$

denote the shift of the standard bi-sided symbolic space

$$\Sigma_k = \{w = (\cdots w_{-1} w_0 w_1 \cdots) \mid w_t \in \mathscr{A},\ t \in \mathbb{Z}\},$$

where $\mathscr{A} = \{0, 1, \ldots, k-1\}$ and $k \geqslant 1$. If W is a σ-invariant closed subset of Σ_k, we call (W, σ) a subshift over $\mathscr{A}$. A function F on W is called *locally constant* if there exists $\ell \in \mathbb{N}$ with

$$F(w) = F(w') \quad \text{whenever } (w_{-\ell} \cdots w_\ell) = (w'_{-\ell} \cdots w'_\ell).$$

Write

$$\mathscr{W}(W) = \{(w_t \cdots w_{t+n-1}) \mid t \in \mathbb{Z},\ n \in \mathbb{N}, w \in W\},$$

called the set of words associated to W. Let

$$[\hat{w}_t \cdots \hat{w}_{t+n-1}] = \{(\cdots w_{-1} w_0 w_1 \cdots) \in W \mid w_t = \hat{w}_t, \ldots, w_{t+n-1} = \hat{w}_{t+n-1}\}$$

for any given word $\hat{w} = (\hat{w}_t \cdots \hat{w}_{t+n-1}) \in \mathscr{W}(W)$, called a block associated to $\hat{w}$.

We easily obtain the following lemma.

Lemma 7.1. *Let (W, σ) be a subshift over $\mathscr{A}$. Then the set of all locally constant cocycles $A : W \to SL(2,\mathbb{R})$ is dense in the space $C^0(W, SL(2,\mathbb{R}))$.*

Proof. Given any $B \in C^0(W, SL(2,\mathbb{R}))$ and any $\varepsilon > 0$. We need only to show that there exists some locally constant $SL(2,\mathbb{R})$-cocycle A with $\operatorname{dist}(A, B) < \varepsilon$. Indeed, by the compactness of W and the continuity of B, there exists some $\ell \in \mathbb{N}$ such that

$$\left|B_{ij}(w) - B_{ij}(w')\right| + \left|B_{ij}^{-1}(w) - B_{ij}^{-1}(w')\right| < \frac{\varepsilon}{4\ell^2} \quad \forall w, w' \in [w_{-\ell} \cdots w_0 \cdots w_\ell]$$

for any word $(w_{-\ell} \cdots w_0 \cdots w_\ell) \in \mathscr{W}(W)$. Now, we assign to any block $[w_{-\ell} \cdots w_\ell]$ a representative, say $\rho \in [w_{-\ell} \cdots w_\ell]$. Put

$$A(w) = B(\rho) \quad \forall w \in [w_{-\ell} \cdots w_\ell].$$

It is easy to see that such A is locally constant such that the requirement.

Thus, the lemma is proved. □

For $\hat{w} = (w_1 \cdots w_{|\hat{w}|}) \in \mathscr{W}(W)$, we write

$$Z_{\hat{w}} = \left\{w \in W \mid (w_1 \cdots w_{|\hat{w}|}) = \hat{w}\right\} = [w_1 \cdots w_{|\hat{w}|}]$$

where $|\hat{w}|$ means the length of the word $\hat{w}$. Finally, if ν is a σ-invariant Borel probability measure on W and $n \in \mathbb{N}$, we set

$$\eta_\nu(n) = \min\left\{\nu(Z_{\hat{w}}) \mid \hat{w} \in \mathscr{W}(W),\ |\hat{w}| = n\right\}.$$

Definition 3. Let (W, σ) be a subshift over $\mathscr{A}$. Then (W, σ) is said to satisfy *condition* (B) if there exists an ergodic probability measure ν on W with

$$\limsup_{n\to\infty} n\eta_\nu(n) > 0.$$

This quite interesting condition was introduced by Boshernitzan in [9]. See [10,42,11,21,20] for some related materials. Specially, in [11] Boshernitzan showed that it implies unique ergodicity for arbitrary minimal subshifts, and in [20] various equivalent characterizations of condition (B) were given.

In this paper, condition (B) enables us to have the following important result, which we need as a lemma in the proof of Theorem 5 below.

Lemma 7.2. *(See [20, Theorem 1].) Let (W, σ) be a minimal subshift that satisfies condition (B). If $A : W \to SL(2,\mathbb{R})$ is locally constant, then A is uniform over (W, σ); that is to say,*

$$\lim_{t\to\infty} \frac{1}{t} \log\left\|A(t, w)\right\| = \lambda(A)$$

holds for every $w \in W$ and uniformly on W.

Note that the above result of Damanik and Lenz together with Lemma 7.1 implies the alternative: the cocycle A, which is either uniformly hyperbolic or uniform subexponential growth, forms a dense subset in $C^0(W, SL(2,\mathbb{R}))$ in the setting of Lemma 7.2. In fact, it forms a residual subset from a recent paper of Avila and Bochi [4]. However, Damanik and Lenz's result still keeps interesting because of it clearly telling us an explicit cocycle (locally constant) which lies in the desired residual set. It is what we are expecting.

7.2. Induced systems and derived cocycles

Let $\nu \in \mathcal{M}_{\mathrm{erg}}(W, \sigma)$ and given any $\ell \in \mathbb{N}$, let

$$Z_\ell = [w_\ell \cdots w_0 \cdots w_\ell] \tag{7.1a}$$

be the block associated to a word $\hat{w} = (w_\ell \cdots w_0 \cdots w_\ell) \in \mathscr{W}(W)$ of length $2\ell + 1$ with

$$0 < \nu(Z_\ell) \leqslant \ell^{-1}. \tag{7.1b}$$

Define

$$\mathfrak{r} : Z_\ell \to \mathbb{N}; \quad w \mapsto \mathfrak{r}(w) = \min\bigl\{t \in \mathbb{N} \bigm| \sigma^t w \in Z_\ell\bigr\}.$$

Note here that Z_ℓ is open–closed in W and (W, σ) is minimal, we see that $\mathfrak{r}(w)$ is well defined for all $w \in Z_\ell$ and locally constant. Thus, we have the homeomorphism

$$\sigma_{z_\ell} : Z_\ell \to Z_\ell; \quad w \mapsto \sigma^{\mathfrak{r}(w)} w,$$

which preserves ergodically $\nu_{z_\ell} := \nu/\nu(Z_\ell)$ and $(Z_\ell, \nu_{z_\ell}, \sigma_{z_\ell})$ is called the *induced system* from (W, ν, σ).

For any $A : W \to SL(2, \mathbb{R})$, we define the derived cocycle

$$A_{Z_\ell} : Z_\ell \to SL(2, \mathbb{R})$$

in the way

$$A_{Z_\ell}(w) = A\bigl(\mathfrak{r}(w), w\bigr) \quad \bigl(= A\bigl(\sigma^{\mathfrak{r}(w)-1} w\bigr) \circ \cdots \circ A(w)\bigr).$$

It is easy to see that $A_{Z_\ell} \in C^0(Z_\ell, SL(2, \mathbb{R}))$ if $A \in C^0(W, SL(2, \mathbb{R}))$, and moreover if A is locally constant then so is A_{Z_ℓ}.

Letting

$$\lambda(A) = \lim_{t \to \infty} \frac{1}{t} \int_W \log\bigl\|A(t, w)\bigr\|\, d\nu \quad \bigl(\text{based on } (W, \sigma)\bigr) \tag{7.2a}$$

and

$$\lambda(A_{Z_\ell}) = \lim_{t \to \infty} \frac{1}{t} \int_{Z_\ell} \log\bigl\|A_{Z_\ell}(t, w)\bigr\|\, d\nu_{z_\ell} \quad \bigl(\text{based on } (Z_\ell, \sigma_{z_\ell})\bigr) \tag{7.2b}$$

we have from [45, Lemma 2.2]

$$\lambda(A_{Z_\ell}) = \lambda(A)/\nu(Z_\ell). \tag{7.3}$$

Write

$$SO(2, \mathbb{R}) = \left\{ R(\phi) = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \,\middle|\, \phi \in \mathbb{R}/(2\pi\mathbb{Z}) \right\}.$$

Then, by a simple computation we have from (7.3)

$$\lambda\big(A \circ R(\phi\chi_{z_\ell})\big) = \nu(Z_\ell)\lambda\big(A_{Z_\ell} \circ R(\phi)\big) \quad \forall \phi \in \mathbb{R}/(2\pi\mathbb{Z}) \tag{7.4}$$

where $\chi_{z_\ell} : W \to \{0, 1\}$ is the characteristic function of Z_ℓ on W.

We will need the following simple lemma.

Lemma 7.3. *Let $A : W \to SL(2, \mathbb{R})$ be continuous and $\varepsilon > 0$. Then, there exists a locally constant $B : W \to SL(2, \mathbb{R})$ such that*

$$\operatorname{dist}(A, B) < \varepsilon \quad \textit{and} \quad B_{Z_\ell}(w) \notin SO(2, \mathbb{R}) \quad \textit{for } w \in Z_\ell.$$

Proof. By virtue of Lemma 7.1, the statement follows easily from the topological structure of Σ_k and the local constantness of $\mathfrak{r}(w)$. □

7.3. Proof of Theorem 5

In what follows, we assume that (W, σ) is a minimal subshift over $\mathscr{A}$ and satisfies condition (B) with ν being the unique σ-invariant Borel probability measure on W. Note that if ν is atomic then Theorem 5 trivially holds. We thus assume from now on that ν is nonatomic. For any $\ell \in \mathbb{N}$, let Z_ℓ be defined as (7.1) in Section 7.2.

By Lemmas 7.2 and 7.3 and Knill's perturbation approaches in [26,18], we can obtain the following, which is equivalent to Theorem 5.

Theorem 7.4. *Let $A : W \to SL(2, \mathbb{R})$ be continuous and $\varepsilon > 0$. Then, there exists some $B \in C^0(W, SL(2, \mathbb{R}))$ which is nonuniformly hyperbolic on W based on (W, σ) such that $\operatorname{dist}(A, B) < \varepsilon$.*

Proof. According to Lemma 7.3, there is no loss of generality in assuming that A is locally constant such that

$$A_{Z_\ell}(w) \notin SO(2, \mathbb{R}) \quad \forall w \in Z_\ell \quad \text{and} \quad \|A\| + \big\|A^{-1}\big\| \leqslant \mathbb{c} \tag{7.5}$$

for any given $\ell \in \mathbb{N}$, where $\mathbb{c} > 0$ is a constant.

Then, it follows at once from a lemma of Herman [25] (see [26, Proposition 2.4]) and the Furstenberg–Kesten ergodic theorem [24] that there is some $\beta \in \mathbb{R}/(2\pi\mathbb{Z})$ such that

$$\lim_{t\to\infty} \frac{1}{t} \log\big\|\big(A_{Z_\ell} \circ R(\beta)\big)(t, w)\big\| = \lambda\big(A_{Z_\ell} \circ R(\beta)\big) > 0 \quad \text{over } (Z_\ell, \nu_{z_\ell}, \sigma_{z_\ell}) \tag{7.6}$$

holds for ν_{z_ℓ}-a.e. $w \in Z_\ell$. Then from (7.4) we have

$$\lambda\big(A \circ R(\beta\chi_{z_\ell})\big) = \nu(Z_\ell)\lambda\big(A_{Z_\ell} \circ R(\beta)\big) > 0 \tag{7.7}$$

and further from Lemma 7.2

$$\lim_{t\to\infty} \frac{1}{t} \log\big\|\big(A \circ R(\beta\chi_{z_\ell})\big)(t, w)\big\| = \lambda\big(A \circ R(\beta\chi_{z_\ell})\big) > 0 \quad \text{over } (W, \nu, \sigma) \tag{7.8}$$

holds uniformly for $w \in W$. Write

$$D = A \circ R(\beta\chi_{z_\ell}) : w \mapsto A(w) \circ R\big(\beta\chi_{z_\ell}(w)\big). \tag{7.9}$$

Therefore, $D : W \to SL(2, \mathbb{R})$ is uniform and is uniformly hyperbolic based on (W, σ) from Lemma 6.11, and there corresponds a continuous, nontrivial, and D-invariant direct decomposition of $\mathbb{R}^2$ into one-dimensional subspaces

$$W \ni w \mapsto \mathbb{E}^u(w) \oplus \mathbb{E}^s(w),$$

where $\mathbb{E}^u$ corresponds the positive exponent $\lambda(D)$ direction. Let $\mathbf{e}_1 = (1, 0)^T \in \mathbb{R}^2$ and let $\sphericalangle(w) = \measuredangle(\mathbf{e}_1, \mathbb{E}^u(w)) \in [0, \pi)$ be the angle between $\mathbf{e}_1$ and $\mathbb{E}^u(w)$ for any $w \in W$. Then the function $\sphericalangle : W \to [0, \pi)$ is continuous. Put

$$P_\mu = R^{-1}(\sphericalangle \circ \sigma) \circ \begin{bmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{bmatrix} \circ R(\sphericalangle \circ \sigma) \quad \forall \mu > 1 \tag{7.10}$$

and

$$\widetilde{B}_\mu = P_\mu \circ D. \tag{7.11}$$

It is evident that the cocycle $\widetilde{B}_\mu$ belongs to $C^0(W, SL(2, \mathbb{R}))$ which has the same continuous $\widetilde{B}_\mu$-invariant direction $\mathbb{E}^u(w)$ as D. By a standard argument we see that

$$\begin{aligned} \lambda(\widetilde{B}_\mu) &= \lim_{t\to\infty} \frac{1}{t} \log\big\|\widetilde{B}_\mu(t, w)\big\| \quad \text{a.e. } w \in W \\ &= \log\mu + \lambda(D) \\ &\geqslant \log\mu > 0. \end{aligned} \tag{7.12}$$

Thus, $\widetilde{B}_\mu$ is nonuniformly hyperbolic for any $\mu > 1$.

From [26, Corollary 2.5] and (7.4) it follows that

$$\mathrm{Leb}\Big(\big\{\beta \in \mathbb{R}/(2\pi\mathbb{Z}) \mid \lambda\big(\widetilde{B}_\mu \circ R(\beta\chi_{z_\ell})\big) = 0\big\}\Big) \leqslant \frac{\nu(Z_\ell)}{\lambda(\widetilde{B}_\mu)} < \frac{1}{\ell\log\mu} \tag{7.13}$$

where $\mathrm{Leb}(\cdot)$ means the Lebesgue measure and where $\lambda(\widetilde{B}_\mu)$ is defined in the manner as in (7.2a). This means that if ℓ is big enough there exists some $\beta_\mu \in \mathbb{R}/(2\pi\mathbb{Z})$ such that

$$0 \leqslant \beta_\mu < \frac{2}{\ell\log\mu} \quad \text{and} \quad \lambda\big(\widetilde{B}_\mu \circ R\big((\beta_\mu - \beta)\chi_{z_\ell}\big)\big) > 0. \tag{7.14}$$

Put

$$B_\mu(\ell) = \widetilde{B}_\mu \circ R\big((\beta_\mu - \beta)\chi_{z_\ell}\big) \quad \text{for } \mu > 1 \text{ and for big } \ell. \tag{7.15}$$

Clearly, $B_\mu(\ell)$ lies in $C^0(W, SL(2, \mathbb{R}))$ for any $\mu > 1$, and it is nonuniformly hyperbolic based on (W, σ) from (7.14). Since

$$\operatorname{dist}(P_\mu, \mathrm{Id}) \leqslant 12\big(\mu - \mu^{-1}\big), \tag{7.16a}$$
$$\|P_\mu\| + \big\|P_\mu^{-1}\big\| \leqslant 2\big(\mu + \mu^{-1}\big) \tag{7.16b}$$

and

$$\operatorname{dist}\big(\mathrm{Id}, R(\beta_\mu)\big) \leqslant 4\beta_\mu \quad \text{if } \ell \text{ is big sufficiently,} \tag{7.16c}$$

we obtain that

$$\operatorname{dist}(P_\mu \circ A, A) \leqslant 12\mathbb{c}\big(\mu - \mu^{-1}\big) \quad \text{by (7.16a)} \tag{7.17}$$

and

$$\begin{aligned}
\operatorname{dist}\big(B_\mu(\ell), P_\mu \circ A\big) &= \operatorname{dist}\big(P_\mu \circ A \circ R(\beta_\mu \chi_{z_\ell}), P_\mu \circ A\big) \\
&\leqslant \|P_\mu \circ A\| \cdot \big\|\big(R(\beta_\mu \chi_{z_\ell}) - \mathrm{Id}\big)\big\| \\
&\quad + \big\|A^{-1} \circ P_\mu^{-1}\big\| \cdot \big\|\big(R(-\beta_\mu \chi_{z_\ell}) - \mathrm{Id}\big)\big\| \\
&\leqslant 32\mathbb{c}\big(\mu + \mu^{-1}\big)\beta_\mu \quad \text{by (7.16b) and (7.16c)} \\
&< \frac{64\mathbb{c}(\mu + \mu^{-1})}{\ell \log \mu} \quad \text{by (7.14).}
\end{aligned} \tag{7.18}$$

Thus, for $\mu > 1$ and for big ℓ we have from (7.17) and (7.18)

$$\begin{aligned}
\operatorname{dist}(B_\mu(\ell), A) &\leqslant \operatorname{dist}\big(B_\mu(\ell), P_\mu \circ A\big) + \operatorname{dist}(P_\mu \circ A, A) \\
&< \frac{64\mathbb{c}(\mu + \mu^{-1})}{\ell \log \mu} + 12\mathbb{c}\big(\mu - \mu^{-1}\big).
\end{aligned} \tag{7.19}$$

In order to prove Theorem 7.4, it is sufficient to take constants $\mu > 1$ and $\ell \in \mathbb{N}$ in the following way:

$$12\mathbb{c}\big(\mu - \mu^{-1}\big) < \varepsilon/3 \quad \text{and} \quad \frac{64\mathbb{c}(\mu + \mu^{-1})}{\ell \log \mu} < \varepsilon/3.$$

Now, $B = B_\mu(\ell)$ satisfies the requirements.

Thus, the proof is completed. □

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Appendix A. A gluing lemma for a pair of maps

In the perturbation theory of dynamical systems, one is frequently confronted with the following context: Suppose that $f : W \to \mathfrak{X}$ is a 'regular' function from a locally compact metric space W into a Banach space $(\mathfrak{X}, |\cdot|)$ and $g : W_1 \to \mathfrak{X}$ a 'regular' function from a compact subset W_1 of W into $\mathfrak{X}$. If g is a small perturbation of the restriction $f|_{W_1}: W_1 \to \mathfrak{X}$, i.e., $\sup_{w\in W_1} |f(w) - g(w)|$ is small sufficiently, then, how to glue g with f to obtain a new 'regular' function $\tilde{f}: W \to \mathfrak{X}$ such that

- $\tilde{f} = g$ on W_1 and
- $\tilde{f}$ is also a small perturbation of f.

When there 'regular' means 'measurable', the function $\tilde{f}$ by simple gluing as follows

$$\tilde{f} = \begin{cases} f & \text{on } W - W_1, \\ g & \text{on } W_1, \end{cases}$$

is measurable satisfying the desired requirements. However, if 'regular' means 'C^r-continuous,' the problem becomes interesting.

In this appendix, by using the Urysohn lemma and the Tietze extension theorem, we obtain the following gluing lemma.

Theorem A.1. *Let W be a locally compact metric space and W_1 a compact subset of W, and let $\varepsilon > 0$ be arbitrarily given. Assume that $\mathcal{C}$ is a closed convex subset of $\mathbb{R}^n$, $1 \leqslant n < \infty$. If $f: W \to \mathcal{C}$ is a continuous function and $g: W_1 \to \mathcal{C}$ is an $\varepsilon/2$-small perturbation of $f|_{W_1}: W_1 \to \mathcal{C}$, then there is a continuous function $\tilde{f}: W \to \mathcal{C}$ which is an ε-small perturbation of f, i.e., $\sup_{x\in W} |f(x) - \tilde{f}(x)| < \varepsilon$, such that $\tilde{f} = g$ on W_1.*

This result is useful in the theory of Lyapunov exponents, specially for linear skew-product systems. Some density results about continuous-time cocycles with nonzero exponents, in the classes of cocycles which are "extremely thin," are often stated under the condition of metric transitivity for the based flow, e.g., [33,36,22]. We indicate that this restrictive hypothesis is not necessary, using Theorem A.1.

Throughout this section, let $C^0(W, \mathrm{gl}(n, \mathbb{R}))$ be the set of all continuous functions $A: W \to \mathrm{gl}(n, \mathbb{R})$ endowed with the C^0-uniform l_1-norm

$$|A| = \sup_{w\in W} \left\{ \sum_{i,j} |A_{ij}(w)| \right\} \quad \forall A \in C^0\big(W, \mathrm{gl}(n, \mathbb{R})\big).$$

Let μ be a Borel probability measure of the Borel measurable space $(W, \mathscr{B})$ and let $\theta: \mathbb{R} \times W \to W$ be any given C^0-flow which is measure-preserving for μ; that is to say, $\theta(0, \cdot) = \mathrm{Id}_W$, $\theta(t+s, \cdot) = \theta(t, \theta(s, \cdot))$ $\forall t, s \in \mathbb{R}$, $\theta(t, w)$, simply written as $t.w$, is jointly continuous in t and w and $\theta(t, \cdot)$ is measure-preserving for μ. Then, for any $A \in C^0(W, \mathrm{gl}(n, \mathbb{R}))$, based on (W, θ) there corresponds a random differential system:

$$\frac{dx}{dt} = A(t.w)x \quad \big((t, x) \in \mathbb{R} \times \mathbb{R}^n;\ w \in W\big).$$

If we let $\Theta_A(t, w): \mathbb{R}^n \to \mathbb{R}^n$ denote the standard fundamental solution matrix of the differential system above, then

$$(\theta, \Theta_A): \mathbb{R} \times W \times \mathbb{R}^n \to W \times \mathbb{R}^n$$

given by

$$(t, w, x) \mapsto \big(t.w, \Theta_A(t, w) \cdot x\big),$$

is the induced smooth linear skew-product flow (or cocycle with continuous time) by the random matrix A based on (W, θ). Conversely, for any given smooth linear skew-product flow $(\theta, \Theta) : \mathbb{R} \times W \times \mathbb{R}^n \to W \times \mathbb{R}^n$ based on (W, θ), by the way

$$A : w \mapsto \frac{d\Theta(t, w)}{dt}\bigg|_{t=0}$$

we obtain a continuous matrix-valued function $A : W \to \mathrm{gl}(n, \mathbb{R})$. Based on θ, the correspondence is one-to-one, so we sometimes identify A with Θ and vice versa.

Assume that μ is ergodic, not necessarily $\operatorname{supp}(\mu) = W$. Then, from the Oseledets multiplicative ergodic theorem [37], we have the Lyapunov spectrum

$$\Sigma_{\mathrm{Lya}}(A, \mu) = \big\{(\lambda_k, n_k) \mid k = 1, \dots, \delta(A, \mu)\big\}.$$

If $\lambda_k \neq 0$ for $1 \leqslant k \leqslant \delta(A, \mu)$, we call $\Sigma_{\mathrm{Lya}}(A, \mu)$ nonuniformly hyperbolic, and if $n_k = 1$ for $1 \leqslant k \leqslant \delta(A, \mu)$, we call $\Sigma_{\mathrm{Lya}}(A, \mu)$ simple. As before, we simply write

$$\lambda_{\max}(A, \mu) = \max\{\lambda_1, \dots, \lambda_\delta\}.$$

We first give two applications before proving Theorem A.1 above.

A.1. Application 1

In [2], Arnold and Cong proved that the $GL(n, \mathbb{R})$-valued cocycles with simple spectrum form a dense subset in $L^\infty(W, GL(n, \mathbb{R}))$ based on an ergodic compact discrete system $\theta : (W, \nu) \to (W, \nu)$ by using the Jordan normal form of a linear cocycle developed by themselves [1]. Moreover, the genericity was also proved by Cong [14] in the L^∞-topology.

As the first application of Theorem A.1, we obtain the following density theorem in the continuous-time case and in the sense of C^0-topology, which is a slight extension of Millionscikov's result (Lemma A.3 below).

Proposition A.2. *Let $(\theta_t)_{t \in \mathbb{R}} : (W, \mu) \to (W, \mu)$ be an ergodic continuous flow of a compact metric space W. If* $\mathrm{Per}(\theta)$*, the totality of all fixed points and periodic points of* (W, θ)*, has μ-measure zero, then the set of A's, which have the simple and nonuniformly hyperbolic spectrum, is dense in* $C^0(W, \mathrm{gl}(n, \mathbb{R}))$ *under the topology of C^0-uniform norm, based on* (W, μ, θ).

Proof. The statement is just a simple combination of Theorem A.1 and the following important result due to Millionscikov.

Lemma A.3. *(See Millionscikov [33].) Let $\theta : \mathbb{R} \times W \to W$ be a continuous flow of a compact metric space W, which preserves an ergodic Borel probability measure ν. Under conditions* $\operatorname{supp}(\nu) = W$ *and* $\nu(\mathrm{Per}(\theta)) = 0$*, the simple spectrum is dense in* $C^0(W, \mathrm{gl}(n, \mathbb{R}))$ *in the sense of C^0-uniform norm topology.*

We next continue the proof of Proposition A.2. For any $A \in C^0(W, \mathrm{gl}(n, \mathbb{R}))$ let $\Sigma_{\mathrm{Lya}}(A, \mu) = \{(\lambda_k, n_k) \mid k = 1, \dots, \delta(A, \mu)\}$. For any $\epsilon \in \mathbb{R}$, let $\epsilon I_n : W \to \mathrm{gl}(n, \mathbb{R})$ be given by $w \mapsto \epsilon I_n$, where I_n is the $(n \times n)$-unit matrix. We can easily get that

$$\Sigma_{\mathrm{Lya}}(A + \epsilon I_n, \mu) = \big\{(\lambda_k + \epsilon, n_k) \mid k = 1, \dots, \delta(A, \mu)\big\}.$$

Therefore, we need only to prove the density of simple spectrum.

Let $A \in C^0(W, \mathrm{gl}(n, \mathbb{R}))$ and $\varepsilon > 0$ be arbitrarily given. As the support of μ, $\mathrm{supp}(\mu)$, is compact and θ-invariant, based on the subsystem $(\mathrm{supp}(\mu), \mu, \theta_{|\mathrm{supp}(\mu)})$, it follows from Lemma A.3 that there is some $B \in C^0(\mathrm{supp}(\mu), \mathrm{gl}(n, \mathbb{R}))$ such that

- $|A(w) - B(w)| < \varepsilon/2 \ \forall w \in \mathrm{supp}(\mu)$;
- B has the simple spectrum.

We then by Theorem A.1 obtain some $\widetilde{A} \in C^0(W, \mathrm{gl}(n, \mathbb{R}))$ such that

- $|A(w) - \widetilde{A}(w)| < \varepsilon \ \forall w \in W$;
- $\widetilde{A} = B$ on $\mathrm{supp}(\mu)$.

Since $\Sigma_{\mathrm{Lya}}(\widetilde{A}, \mu) = \Sigma_{\mathrm{Lya}}(B, \mu)$, $\widetilde{A}$ has the simple spectrum based on (W, μ, θ).

From the arbitrariness of ε we get the desired density. $\square$

A.2. Application 2

Let

$$\mathrm{sl}(2, \mathbb{R}) = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} \,\middle|\, \alpha, \beta, \gamma \in \mathbb{R} \right\}.$$

For any $H \in C^0(W, \mathrm{sl}(2, \mathbb{R})) \subset C^0(W, \mathrm{gl}(2, \mathbb{R}))$

$$\frac{dx}{dt} = H(t.w)x, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, w \in W,$$

is a random linear Hamiltonian differential equation. For any given closed convex subset I of $\mathbb{R}$ and for any $\alpha \in \mathbb{R}$, put

$$\mathrm{sl}_{I,\alpha}(2, \mathbb{R}) = \left\{ \begin{bmatrix} 0 & 1 \\ r & 0 \end{bmatrix} \,\middle|\, r \in I - \alpha \right\},$$

$$\mathrm{sl}_{I}(2, \mathbb{R}) = \left\{ \begin{bmatrix} 0 & 1 + r \\ 1 - r & 0 \end{bmatrix} \,\middle|\, r \in I \right\}.$$

Then, for any $S \in C^0(W, \mathrm{sl}_{I,\alpha}(2, \mathbb{R}))$

$$\frac{dx}{dt} = S(t.w)x, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \ w \in W,$$

is a random one-dimensional Schrödinger equation based on (W, θ). For any $V \in C^0(W, \mathrm{sl}_I(2, \mathbb{R}))$

$$\frac{dx}{dt} = V(t.w)x, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, w \in W,$$

is a random Vinograd equation based on (W, θ) [43].

The following slight extension of Nerurkar's theorem ([36, Theorem 1.8]) is another application of Theorem A.1.

Proposition A.4. *Let (W, θ, μ) be a measure-preserving C^0-flow over a compact metric space W. Let S be a subset of* $\mathrm{sl}(2, \mathbb{R})$. *Assume that*

(a) *μ is an ergodic probability measure, not necessarily* $\mathrm{supp}(\mu) = W$, *and μ is not supported on a single orbit,*
(b) *S is closed and convex in* $\mathrm{sl}(2, \mathbb{R})$,
(c) *S has the strong accessibility property.*

Let

$$C^{\mu}_{\mathrm{pos}}(W, S) = \left\{A \in C^0(W, S) \;\middle|\; \lambda_{\max}(A, \mu) > 0\right\}.$$

Then, $C^{\mu}_{\mathrm{pos}}(W, S)$ is a dense subset of $C^0(W, S)$ under the topology of C^0-uniform norm.

In particular, we obtain the following

Corollary A.5. *Let (W, μ, θ) be an ergodic flow of a compact metric space W and I be a closed, convex, and nonsingleton subset of $\mathbb{R}$, and let $\alpha \in \mathbb{R}$. If μ is not supported on a single orbit, then, based on (W, μ, θ), we have the following three statements*:

(1) *The nonuniform hyperbolicity is dense in $C^0(W, \mathrm{sl}(2, \mathbb{R}))$ under the topology of C^0-uniform norm [27].*
(2) *The nonuniform hyperbolicity is dense in $C^0(W, \mathrm{sl}_{I,\alpha}(2, \mathbb{R}))$ under the topology of C^0-uniform norm.*
(3) *The nonuniform hyperbolicity is dense in $C^0(W, \mathrm{sl}_I(2, \mathbb{R}))$ under the topology of C^0-uniform norm.*

We first recall the Nerurkar theorem. As usual, $\mathrm{sl}(2, \mathbb{R})$ is a Lie algebra of $SL(2, \mathbb{R})$. Let S be a set of $\mathrm{sl}(2, \mathbb{R})$. Let $L(S)$ be the Lie subalgebra of $\mathrm{sl}(2, \mathbb{R})$ generated by S, and let $L_0(S)$ denote the ideal of $L(S)$ generated by the difference $S - S = \{x - y \mid x \in S, y \in S\}$. S is said to have the *strong accessibility* provided that $L_0(S) = \mathrm{sl}(2, \mathbb{R})$. Clearly, $\mathrm{sl}_{I,\alpha}(2, \mathbb{R})$ and $\mathrm{sl}_I(2, \mathbb{R})$ both have the strong accessibility property; see [36, Examples 1 and 2]. This shows Corollary A.5 from Proposition A.4.

The following important result is due to M. Nerurkar.

Lemma A.6. *(See [36, Theorem 1.8].) Let (W, θ, μ) be a measure-preserving C^0-flow over a compact metric space W. Let S be a subset of* $\mathrm{sl}(2, \mathbb{R})$. *Assume that*

(a) *μ is an ergodic probability measure such that* $\operatorname{supp}(\mu) = W$ *and μ is not supported on a single orbit,*
(b) *S is closed and convex in* $\mathrm{sl}(2, \mathbb{R})$*,*
(c) *S has the strong accessibility property.*

Let

$$C_{\mathrm{pos}}^{\mu}(W, S) = \left\{A \in C^0(W, S) \,\middle|\, \lambda_{\max}(A, \mu) > 0\right\}.$$

Then, $C_{\mathrm{pos}}^{\mu}(W, S)$ is a dense subset of $C^0(W, S)$ under the topology of C^0-uniform norm.

We are now in a position to prove Proposition A.4.

Proof of Proposition A.4. In order to prove Proposition A.4, we need only to show that for any $H \in C^0(W, S)$ and for any $\varepsilon > 0$, there is some $\widetilde{H} \in C^0(W, S)$ such that

- $|H - \widetilde{H}| < \varepsilon$ and
- $\lambda_{\max}(\widetilde{H}, \mu) > 0$.

Since the restriction of H on $\operatorname{supp}(\mu)$,

$$H_{|\operatorname{supp}(\mu)} : \operatorname{supp}(\mu) \to S,$$

is in $C^0(\operatorname{supp}(\mu), S)$, by Lemma A.6 we can find some $B \in C^0(\operatorname{supp}(\mu), S)$ such that

$$\left|H_{|\operatorname{supp}(\mu)} - B\right| < \varepsilon/2 \quad \text{on } \operatorname{supp}(\mu) \text{ and } \lambda_{\max}(B, \mu) > 0$$

based on the subsystem $(\operatorname{supp}(\mu), \theta, \mu)$. Then, from Theorem A.1 we obtain that there is a continuous function

$$\widetilde{H} : W \to S$$

such that

$$\widetilde{H}(w) = B(w) \quad \text{for } w \in \operatorname{supp}(\mu)$$

and

$$|H - \widetilde{H}| < \varepsilon \quad \text{on } W.$$

It is obvious that $\lambda_{\max}(\widetilde{H}, \mu) = \lambda_{\max}(B, \mu) > 0$.

The proof of Proposition A.4 is thus complete. □

A.3. Proof of Theorem A.1

In order to prove Theorem A.1, we will need a lemma.

Lemma A.7. *Let $\mathcal{C}$ be a closed convex subset of a Hilbert space $\mathscr{H}$. Then, $\mathcal{C}$ is a retract of $\mathscr{H}$; that is to say, there is a continuous retraction $r:\mathscr{H}\to\mathcal{C}$ such that $r(c)=c$ for all $c\in\mathcal{C}$.*

Proof. For any $x\in\mathscr{H}$ there is a unique $c(x)$ in $\mathcal{C}$ such that

$$\big|x-c(x)\big|=\inf_{c\in\mathcal{C}}|x-c|.$$

Let $r:\mathscr{H}\ni x\mapsto c(x)\in\mathcal{C}$. It is obvious that r is a retraction of $\mathscr{H}$ onto $\mathcal{C}$. □

Proof of Theorem A.1. Let $f:W\to\mathcal{C}$ and $g:W_1\to\mathcal{C}$ and ε all be given as in Theorem A.1. It follows from Lemma A.7 that $\mathcal{C}$ is a retract of $\mathbb{R}^n$. Since W is a metric space, it is normal and thus the triple $(W,W_1;\mathcal{C})$ has the universal extension property and then there is a continuous function $\tilde{g}:W\to\mathcal{C}$ which is an extension of $g:W_1\to\mathcal{C}$ from the Tietze extension theorem [34, Theorem 35.1 or Exercises §35.5].

Since $f,\tilde{g}:W\to\mathcal{C}$ both are continuous and W_1 is compact, there is some $\delta>0$ such that for any $w\in W_1$ we have

$$\big|f(w)-f(w')\big|<\frac{\varepsilon}{4}\quad\text{if }\operatorname{dist}(w,w')<\delta,$$
$$\big|g(w)-\tilde{g}(w')\big|<\frac{\varepsilon}{4}\quad\text{if }\operatorname{dist}(w,w')<\delta.$$

Let $B_\delta(W_1)=\{w\in W\mid \operatorname{dist}(W_1,w)<\delta\}$. Then, by the Urysohn lemma [34, Theorem 33.1], there is a continuous function

$$\eta:W\to[0,1]$$

such that $\eta\equiv1$ on W_1 and $\eta\equiv0$ in $W_1-B_\delta(W_1)$.

We now define

$$\tilde{f}:W\to\mathcal{C}$$

in the way

$$\tilde{f}(w)=\big[1-\eta(w)\big]f(w)+\eta(w)\tilde{g}(w)\quad\forall w\in W.$$

It is easily seen that $\tilde{f}=g$ on W_1 and $\tilde{f}=f$ on $W-B_\delta(W_1)$. We next need only to show

$$\big|\tilde{f}(w)-f(w)\big|<\varepsilon\quad\text{for any }w\in B_\delta(W_1)-W_1.$$

In fact, to any $w \in B_\delta(W_1) - W_1$, there corresponds to some $x_w \in W_1$ satisfying $\mathrm{dist}(w, x_w) < \delta$. Thus

$$\begin{aligned}\left|\tilde{f}(w) - f(w)\right| &= \left|-\eta(w) f(w) + \eta(w)\tilde{g}(w)\right| \\ &\leqslant \left|-f(w) + f(x_w) - f(x_w) + \tilde{g}(w) - g(x_w) + g(x_w)\right| \\ &< \varepsilon,\end{aligned}$$

as desired.

The proof of Theorem A.1 is thus completed. □

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